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CONTIGUOUS RELATIONS AND RELATED FORMULAS FOR THE H-FUNCTION OF FOX

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Abstract. The set of contiguous relations for the H-function of Fox are obtained by a unified treatment in a direct and simple manner, the connections among these formulas are discussed, and the connections with some scattered results in the literature are given. Several finite series involving H-functions can be generated from each of these contiguous relations; some examples are included for illustration.

1. Introduction. The H-function was introduced by C. Fox [5] and it is usually defined in terms of the contour integral

$$H_{p,q}^{m,n}\left[z \;\middle|\; \begin{matrix}\{(a_i, \alpha_i)\}\\ \{(b_i, \beta_i)\}\end{matrix}\right]=$$

$$=\frac{1}{2\pi i}\int_C \frac{\prod_{i=1}^{n}\Gamma(1-a_i+\alpha_i s)\prod_{i=1}^{m}\Gamma(b_i-\beta_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i-\alpha_i s)\prod_{i=m+1}^{q}\Gamma(1-b_i+\beta_i s)}\, z^s\, ds,$$

in which we use the notations $\{(a_i, \alpha_i)\}$ and $\{(b_i, \beta_i)\}$ to denote, respectively, the sets of parameters $(a_1, \alpha_1), \ldots, (a_p, \alpha_p)$ and $(b_1, \beta_1), \ldots, (b_q, \beta_q)$. The description of the path of integration and the conditions which the parameters must satisfy are described in [5]. An extension of the definition is given in [14]. If the second parameter of every pair is equal to 1, then the H-function reduces to the G-function of Meijer, a summary of the properties of which appear in the books [4] and [11].

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A number of recurrence formulas for the H-function have appeared, scattered in recent literature. Some of these can be given the name of contiguous relations; where they involve a unit shift in any of the first parameters of the pairs, the terminology thus corresponds to that used for the Gauss hypergeometric function. Various methods have been used in order to derive such formulas; for example, see [1], [2], [3], [6], [7], [8], [9], [10], [12], and [15]. Often the method used involves the evaluation of a complicated integral of a product of the H-function and some other function and the use of an identity for the second function. Many of the results contain restrictive conditions on the second parameters of the pairs. In section 2 we collect the 30 contiguous relations. In a unified treatment they are obtained by simple direct derivations and without the severe restrictions on the second parameters. Further, the connections with certain other formulas which have appeared in the literature are discussed.

Finite series which involve the H-function also have appeared in a number of places, for example, in [3], [13], and [15]. Similar methods were used and often with severe restrictions placed upon the second parameters of the pairs. In section 3 we discuss the manner in which various simple forms of finite series can be obtained directly from our contiguous relations and how the complicated generalizations of some known series can also be obtained, again with fewer restrictions on the parameters.

Inasmuch as certain determinants appear throughout this work, we shall introduce simplifying notations as, for example

$$d(b_1, a_p-k)=\det\begin{bmatrix} b_1 & a_p-k \\ \beta_1 & \alpha_p \end{bmatrix},$$

in which we display the first row of the determinant by our notation. The second row of the determinant is always to be filled in with the appropriate α,s and β,s in order to correspond to those a,s and b,s of the first row. We assume that none of these determinants equals zero, which is consistent with the restrictions in the definition of the H-function. To further simplify the notational problems we merely write H for the function as given in the defining relation; then, for example, we use $H[b_1+1]$ to denote the contiguous function in which b_1 is replaced by b_1+1, but with all other parameters left unchanged, where further it is to be understood that if both H and $H[b_1+1]$ appear, then $m \geqq 1$.

2. Contiguous relations. We first note from the definition that the replacement of b_1 by b_1+1 and the application of the recurrence formula $\Gamma(z+1)=z\Gamma(z)$ is equivalent to the introduction of the multipiler $b_1-\beta_1 s$ into the contour integral format for H. Similarlv, the replacement of b_q by b_q+1 introduces $-b_q+\beta_q s$, of a_1 by a_1-1 introduces $1-a_1+\alpha_1 s$, and of a_p by a_p-1 introduces $a_p-1-\alpha_p s$. Consequently, we can simply form a 3–term recurrence involving undetermined coefficients.

$$AH[b_1+1]+BH[a_p-1]=CH$$

and then require that

$$A(b_1-\beta_1 s)+B(a_p-1-\alpha_p s)=C$$

be an identity in s. Hence A, B and C can be evaluated in order to obtain the contiguous relation numbered (1) in our list. In a similar manner all of the other listed contiguous relations can be obtained.

(1) $\alpha_p H[b_1+1]-\beta_1 H[a_p-1]=d(b_1,a_p-1)H$

(2) $\alpha_p H[a_1-1]+\alpha_1 H[a_p-1]=-d(a_1-1,a_p-1)H$

(3) $\beta_q H[a_1-1]-\alpha_1 H[b_q+1]=-d(a_1-1,b_q)H$

(4) $\beta_q H[b_1+1]+\beta_1 H[b_q+1]=d(b_1,b_q)H$

(5) $\alpha_1 H[b_1+1]+\beta_1 H[a_1-1]=d(b_1,a_1-1)H$

(6) $\beta_q H[a_p-1]+\alpha_p H[b_q+1]=d(a_p-1,b_q)H$

(7) $\beta_2 H[b_1+1]-\beta_1 H[b_2+1]=d(b_1,b_2)H.$

(8) $\alpha_2 H[a_1-1]-\alpha_1 H[a_2-1]=-d(a_1-1,a_2-1)H$

(9) $\alpha_{p-1} H[a_p-1]-\alpha_p H[a_{p-1}-1]=d(a_p-1,a_{p-1}-1)H$

(10) $\beta_{q-1} H[b_q+1]-\beta_q H[b_{q-1}+1]=-d(b_q,b_{q-1})H$

(11) $d(a_p-1,b_q)H[a_1-1]-d(b_q,a_1-1)H[a_p-1]=-d(a_1-1,a_p-1)H[b_q+1]$

(12) $d(a_p-1,b_q)H[b_1+1]+d(b_q,b_1)H[a_p-1]=d(b_1,a_p-1)H[b_q+1]$

(13) $d(a_1-1,b_q)H[b_1+1]-d(b_q,b_1)H[a_1-1]=d(b_1,a_1-1)H[b_q+1]$

(14) $d(a_1-1, a_p-1)\, H[b_1+1] - d(a_p-1, b_1)\, H[a_1-1] = -d(b_1, a_1-1)\, H[a_p-1]$

(15) $d(b_2, b_3)\, H[b_1+1] + d(b_3, b_1)\, H[b_2+1] = -d(b_1, b_2)\, H[b_3+1]$

(16) $d(a_2-1, a_3-1)\, H[a_1-1] + d(a_3-1, a_2-1)\, H[a_2-1] = -d(a_1-1, a_2-1)\, H[a_3-1]$

(17) $d(a_{p-1}-1, a_{p-2}-1)\, H[a_p-1] + d(a_{p-2}-1, a_p-1)\, H[a_{p-1}-1] = -d(a_p-1, a_{p-1}-1)\, H[a_{p-2}-1]$

(18) $d(b_{q-1}, b_{q-2})\, H[b_q+1] + d(b_{q-2}, b_q)\, H[b_{q-1}+1] = -d(b_q, b_{q-1})\, H[b_{q-2}+1]$

(19) $d(a_p-1, b_1)\, H[a_{p-1}-1] + d(b_1, a_{p-1}-1)\, H[a_p-1] = -d(a_{p-1}-1, a_p-1)\, H[b_1+1]$

(20) $d(b_q, a_1-1)\, H[b_{q-1}+1] + d(a_1-1, b_{q-1}) H[b_q+1] = -d(b_{q-1}, b_q) H[a_1-1]$

(21) $d(a_2-1, a_p-1) H[a_1-1] + d(a_p-1, a_1-1) H[a_2-1] = d(a_1-1, a_2-1)\, H[a_p-1]$

(22) $d(b_2, b_q) H[b_1+1] + d(b_q, b_1) H[b_2+1] = d(b_1, b_2) H[b_q+1]$

(23) $d(a_{p-1}-1, a_1-1) H[a_p-1] + d(a_1-1, a_p-1) H[a_{p-1}-1] = d(a_p-1, a_{p-1}-1) H[a_1-1]$

(24) $d(b_{q-1}, b_1) H[b_q+1] + d(b_1, b_q) H[b_{q-1}+1] = d(b_q, b_{q-1}) H[b_1+1]$

(25) $d(a_2-1, b_q) H[a_1-1] + d(b_q, a_1-1) H[a_2-1] = -d(a_1-1, a_2-1) H[b_q+1]$

(26) $d(b_2, a_p-1) H[b_1+1] + d(a_p-1, b_1) H[b_2+1] = -d(b_1, b_2) H[a_p-1]$

(27) $d(a_2-1, b_1) H[a_1-1] + d(b_1, a_1-1) H[a_2-1] = d(a_1-1, a_2-1) H[b_1+1]$

(28) $d(b_2, a_1-1) H[b_1+1] + d(a_1-1, b_1) H[b_2+1] = d(b_1, b_2) H[a_1-1]$

(29) $d(a_{p-1}-1, b_q) H[a_p-1] + d(b_q, a_p-1) H[a_{p-1}-1] = d(a_p-1, a_{p-1}-1)\, H[b_q+1]$

(30) $d(b_{q-1}, a_p-1) H[b_q+1] + d(a_p-1, b_q) H[b_{q-1}+1] = d(b_q, b_{q-1}) H[a_p-1]$

Alternatively, it should be noted that many of these relations are closely connected. The H-function satisfies the identitv

$$H^{m,n}_{p,q}\left[z\,\Big|\,\begin{matrix}\{a_i,\ \alpha_i)\}\\ \{b_i,\ \beta_i)\}\end{matrix}\right] = H^{n,m}_{q,p}\left[z^{-1}\,\Big|\,\begin{matrix}\{(1-b_i,\ \beta_i)\}\\ \{(1-a_i,\ \alpha_i)\}\end{matrix}\right]$$

so that, for example, (3) and (4) can be obtained, respectively, from (1) and (2) and beyond formula (6) each odd numbered formula leads in the same manner to the even numbered successor. Actually we could consider formulas (1) and (2) as basic and then derive all of the others from the two of them along with this transformation formula. For example, (5) can be obtained by combining (1) and (2) and similarly (6) from (2) and (3). A double application of (1) leads to (7) and to (9), (11) comes from (2) and (3), (13) from (3) and (4), (15) from a triple application of (7), and similarly (17) from (9). Formula (19) can be obtained by a double application of (1) and similarly (21), (23), (25), (27), and (29) from (2), (2), (3), (5), and (6), respectively.

If we set all of the α's and β's equal to 1 we obtain the special cases for the G-function of Meijer, where now, for example, $d\ (b_1,\ a_p-k)$ is simply the difference, $b_1-(a_p-k)$. Since MacRoberts' E-function and the generalized hypergeometric series ${}_pF_q$ are special cases of the G-function, some contiguous relations for these functions now also follow.

Some of the formulas have appeared in various format and are scattered in the literature; some examples which are known to us follow. Formulas (1)-(6) were earlier derived by P. Anandani [2] from differentiation formulas and without unnecessary restrictions on the α's and β's; cases of (1)—(4) and (6), but with the restriction of equality of those α's and β's related to the a's and b's of the contiguity of the particular formula, were also given by P. Anandani [3] as special cases of certain finite sums. Formulas related to (5) and (6) were obtained from differentiation formulas by B. M. Agrawal [1]. Formula (2) has been derived using integrals involving the identities connecting generalized Bessel functions by Aruna Srivastava and K. C. Gupta [15]; cases which follow directly from formulas (3), (8) and (25) were given earlier by K. C. Gupta [7]. U. C. Jain [10] gives a special case of (28) with $\alpha_1 = \beta_1 = \beta_2$; P.C. Golas [6] obtains the special case $\alpha_1=\alpha_p=\beta_1=\beta_2$ of formula (1) from integrals and identities involving the Gauss hypergeometric function. A result which is a simple combination of (5) and (14), but with the restriction $\alpha_1=\alpha_p=\beta_1$ was obtained from integrals and identities involving Laguerre polynomials by S. L Mathur [12]. Other recurrences of four and more terms which have appeared can be obtained from combinations of our contiguous relations, such as those in [6] and [12].

Relations analogous to those for the Gauss hypergeometric function in which the coefficients are polynomials of degree one in z are not available for the general H-function, since from

$$zH^{m,n}_{p,q}\left[z \;\Big|\; \begin{matrix}\{(a_i, \alpha_i)\}\\ \{(b_i, \beta_i)\}\end{matrix}\right] = H^{m,n}_{p,q}\left[z \;\Big|\; \begin{matrix}\{(a_i+\alpha_i, \alpha_i)\}\\ \{(b_i+\beta_i, \beta_i)\}\end{matrix}\right]$$

it is seen that contiguous type functions appear only for the special case of the G-function.

3. **Finite Series.** Certain finite series can be obtained from the contiguous relations by the formation of collapsing series. We illustrate for the case of formula (1), the others can be obtained, although beyond formula (10) the coefficients become quite messy in comparison with their first formulas. If we pair the terms which involve a_p and first write

$$\beta_1\, H\,[b_1+k-1,\, a_p-1] = \alpha_p\, H\,[b_1+k,\, a_p] - d\,(b_1+k-1,\, a_p-1)\, H\,[b_1+k-1,\, a_p]$$

and then note that

$$d\,(b_1+k-1,\, a_p-1) = \alpha_p\, \Gamma\,(b_1+k-(a_p-1)\,\beta_1/\alpha_p)/\Gamma\,(b_1+k-1-(a_p-1)\,\beta_1/\alpha_p)$$

we can sum on k and collapse the resulting series on the right. Consequently, we obtain

$$\text{(1a)}\quad \frac{\beta_1}{\alpha_p}\,\Sigma^n_{k=1}\frac{H\,[b_1+k-1,\, a_p-1]}{\Gamma\,(b_1+k-(a_p-1)\,\beta_1/\alpha_p)} = \frac{H\,[b_1+n,\, a_p]}{\Gamma\,(b_1+n-(a_p-1)\,\beta_1/\alpha_p)} - \frac{H\,[b_1,\, a_p]}{\Gamma\,(b_1-(a_p-1)\,\beta_1/\alpha_p)}.$$

If we similarly begin with the form

$$\alpha_p\, H\,[b_1+1,\, a_p-k+1] = \beta_1\, H\,[b_1,\, a_p-k] - d\,(b_1,\, a_p-k)\, H\,[b_1,\, a_p-k+1]$$

(here it seems more convenient to decrease the indices, since one of the a's is involved), then a similar formula can be derived,

$$\text{(1b)}\quad \frac{\alpha_p}{\beta_1}\,\Sigma^n_{k=1}\frac{(-1)^k\, H\,[b_1+1,\, a_p-k+1]}{\Gamma\,(b_1\,\alpha_p/\beta_1-(a_p-1)+k)} = \frac{(-1)^n\, H\,[b_1,\, a_p-n]}{\Gamma\,(b_1\,\alpha_p/\beta_1-(a_p-1)+n)} - \frac{H\,[b_1,\, a_p]}{\Gamma(b_1\,\alpha_p/\beta_1-(a_p-1))}.$$

The third pairing of terms in the form

$$d\,(b_1+k,\, a_p+k-1)\; H[b_1+k,\, a_p+k] = \beta_1\; H[b_1+k,\, a_p+k-1] - \alpha_p\, H\,[b_1+k-1,\, a_p+k]$$

leads to a series in which both parameters are involved in the summation,

$$\text{(1c)}\quad \Sigma_{k=1}^{n}\ \alpha_p^{k-1}\ \beta_1^{\,n-k}\ d\,(b_1+k,\ a_p+k-1)\ \ H\,[b_1+k,\ a_p+k]$$
$$= \alpha_p^n\,H\,[b_1+n+1,\ a_p+n] - \beta_1^n\ H\,[\,b_1+1,\ a_p\,].$$

P. N. Rathie [13] has obtained a formula of this third type for the G-function which would be generalized by starting with our formula (2).

The summation formulas given by P. Anandani [3] are not included in these three types. They can be obtained, and without the restrictions on the second parameters of the pairs, but the coefficients are extremely messy and no simplifying notation is obvious to us at this time. In order to do this we consider formula (1) written in the format

$$d(b_1,\ a_p-1)H[b_1,\ a_p-1]=\alpha_p H[b_1+1,\ a_p]-\beta_1 H[b_1,a_p-1]\ .$$

If we now iterate by expanding each term on the right by use of this same relation, we obtain

$$d(b_1+1,\ a_p-1)d(b_1,\ a_p-1)d(b_1,\ a_p-2)H[b_1,\ a_p]=$$

$$=\alpha_p^2 d(b_1,\ a_p-2)H[b_1+2,\ a_p]-$$

$$-\alpha_p\beta_1\Big(d(b_1+1,\ a_p-1)+d(b_1,\ a_p-2)\Big)H[b_1+1,\ a_p-1]+$$

$$+\ \beta_1^2 d(b_1+1,\ a_p-1)H[b_1,\ a_p-2]\ .$$

Further iterations produce expressions of the form

$$\text{(1d)}\qquad A_0H[b_1,\ a_p]=\ \Sigma_{k=0}^{n}\ A_kH[b_1+n-k,\ a_p-k]$$

in which the A's involve sums and products of the determinants. For the special case in which $\beta_1=\alpha_p$ the coefficients are greatly simplified and we can write

$$\Big(\Gamma(b_1-a_p+n)/\Gamma(b_1-a_p+1)\Big)H[b_1,\ a_p]=$$
$$=\ \Sigma_{k=0}^{n}(-1)^k\binom{n}{k}H[b_1+n-k,\ a_p-k]$$

which is thus of the same type as certain of the summations in [3].

Related finite series for the G-function and in suitable cases for the E and ${}_pF_q$ functions can now be obtained by specializing the parameters.

In various papers recurrences have appeared in which the indices m,n,p,q have not been the same throughout; we have here intentionally omitted the discussion of such formulas.

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