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# GRAVITATIONAL INSTABILITY OF AN INFINITELY EXTENDING VISCOUS LAYER SURROUNDED BY NON-CONDUCTING MATTER

By

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## ABSTRACT

We have studied the magneto-gravitational instability of a rotating infinite fluid layer of finite thickness. The layer is surrounded by a rotating, infinitely extending non-conducting matter. The dispersion relation for equal kinematic viscosities has been derived and studied in various situations. The cases of short and long wave lengths have also been discussed. It is observed that the rotation has no effect on the equilibrium if the viscosity is predominant. In case of short waves also rotation plays no role.

## 1. INTRODUCTION

The importance of the study of gravitational and magneto-gravitational stability of fluid layers of finite thickness in the astronomical context was pointed out by Shafranow[1]. Ognesyan[2,3] studied the same in the presence of a uniform magnetic field and Chakraborty[4] extended his investigations to study the effeet of uniform rotation. Uberoi[5] considered the same problem in the presence of infinitely extending surrounding non-conducting fluid on both sides of the layers. Srivastava and Sharma[7] extended the problem of Uberoi to consider the effect of finite conductivity. The aim of the present paper is to extend our later investigations to include the effect of unform rotation of both the layer and the surrounding matter.

## 2. LINEARIZED EQUATIONS AND THEIR SOLUTIONS.

Consider an infinitely extending plane layer of viscous, gravitating and ideally conducting fluid mass of uniform density. The thickness of the layer is taken to be *2h*. It is supposed that the xoy plane coincides with the equilibrium middle level of the layer. The z-axis is directed upwards and is normal to the unperturbed fluid surfaces. This layer is surrounded by a non-conducting fluid of constant density $\rho_0$. The entire matter is incompressible.

In the static state the conducting fluid is supposed to be immersed in an external uniform magnetic field of strength $\bar{H}_0$ along the x-axis. The layer and the surrounding liquid both are assumed to be rotating about z-axis with uniform angular velocity.

The linearised hydro-magnetic equations for the conducting layer governing the small departures from equilibrium are Equations of momentum.

$$(2.1) \quad \frac{\partial \bar{u}}{\partial t} + 2\bar{\Omega} \, x \bar{v} = -\nabla \pi_1 + \frac{1}{4\pi\rho} \, (\bar{H}_0 . \nabla) \, \bar{h} + \nu \nabla^2 \bar{u} \, ,$$

$$(2.2) \quad \pi_1 = \frac{\delta p}{\rho} + \delta v + \frac{\bar{H}_0 . \bar{h}}{4\pi\rho} \, ,$$

Equation of continuity :

$$(2.3) \quad \text{div} \, \bar{v} = 0,$$

Maxwell's equations :

$$(2.4) \quad \text{div} \, \bar{h}_l = 0,$$

$$(2.5) \quad \frac{\delta h}{\delta t} = \text{curl} \, (\bar{v} \, x \, \bar{H}_0) \, , \quad \bar{H}_0 = [H_0, 0, 0],$$

Poisson's Equation :

$$(2.6) \quad \nabla^2 \delta V_l = 0 \, ,$$

where $\delta p$, $\bar{v}$ , $\bar{h}_l$ , $\delta V_l$ denote the perturbations in velocity field (initially assumed to be zero), magnetic field and gravitational potential.

With the same notations as above the corresponding linearized equations for the non-conducting medium are,

$$(2.7) \quad \frac{\delta v}{\delta t} + 2 \, \bar{\Omega} \, x \, \bar{u} = - \nabla \pi_e + \nu_0 \nabla^2 \, \bar{u} \, ,$$

(2.8) $\pi_e = \frac{\delta p_e}{\rho} + \delta v$ ,

(2.9, 10) $\operatorname{div} \bar{u} = 0$ , $\operatorname{div} \bar{h}_e = 0$ ,

(2.11, 12) $\operatorname{curl} \bar{h}_e = 0$ , $\nabla^2 \delta v_e = 0$,

where, now $\delta p_e$ , $\bar{u}$ , $\bar{h}_e$ and $\delta v_e$ denote corresponding perturbations in the matter surrounding the layer.

To study the stability of the system, it is assumed that all the physical quantities vary as

(2.13) $F(x, y, z, t) = f(z) \exp. [ (\sigma t + \overline{ik_1 x + k_2 y}) ]$.

Equation (2.5) can be written as

(2.14) $\bar{h} = \frac{ik_1 H_0}{\sigma} \bar{v}$ .

Substituting for $\bar{h}$ from (2.14) in (2.1) and taking its divergence, we have

(2.15) $\nabla^2 \pi_i = 2 \bar{\Omega} \nabla \times \bar{v} = 2 \Omega \zeta_i$,

where $\zeta_i$ is the z component of the vorticity, given by

(2.16) $\zeta_i = ik_1 v_y - ik_2 v_x$ .

The equation (2.1) can be written as

(2.17) $(A - \nu \nabla^2) \bar{v} + 2 \bar{\Omega} \times \bar{v} = -\nabla \pi_i$ .

From equation (2.13), we have

(2.18) $\frac{\delta^2}{\delta x^2} = - k_1^2 , \frac{\delta^2}{\delta y^2} = - k_2^2 , \frac{\delta}{\delta t} = \sigma$ .

Equation (2.17) breaks up into three component equations

(2.19) $(B - \nu D^2) v_x - 2 \Omega v_y = - ik_1 \pi_i$

(2.20) $(B - \nu D^2) v_y - 2 \Omega v_x = - ik_2 \pi_i$

(2.21) $(B-\nu D^2)\, v_z = -D\pi_1$

and the equation of continuity gives

(2.22) $ik_1 v_x + ik_2 v_y + D v_z = 0$

where $B = A + \nu\,(k_1^2 + k_2^2) = A + \nu k^2$

(2.23) $A = \left(\sigma + \frac{\Omega^2 A}{\sigma}\right), \quad \Omega^2 A = \frac{u k_1^2 H_o^2}{4\pi\rho}$

From equations (2.15, 2.16 and 2.19 — 2.22) we have

(2.24) $(D^2-m^2)\, v_z = 1/\nu\, D\, \pi_1$

(2.25) $(D^2-k^2)\, \pi_1 = 2\,\Omega\, \zeta_1$

(2.26) $(D^2-m^2)\, Dv_z - \frac{2\Omega}{\nu}\, \zeta_1 = \frac{k^2}{\nu}\, \pi_1$

(2.27) $(D^2-m^2)\, \zeta_1 - \frac{2\,\Omega}{\nu}\, Dv_z = 0$

(2.28) where $m^2 = B/\nu$.

From equations (2.24 — 2.27) we obtain

(2.29) $\left[(D^2-k^2)\,(D^2-m^2)^2 - \frac{4\Omega^2}{\nu^2}\, D^2\right] \pi_1 = 0\,.$

The solution of (2.29), considering the symmetry of $u_z$, is written as

(2.30) $\pi_1 = C_1 \sinh pz.$

(2.30a) where p is a positive root of $(p^2-k^2)\,(p^2-m^2)^2 - \frac{4\Omega^2}{\nu^2}\, p^2 = 0.$

From equations (2.24) and (2.30) the solution for $v_z$ is written as

(2. 1) $v_z = C_2 \cosh mz + \frac{p\, C_1 \cosh pz}{\nu\,(p^2-m^2)}$ ,

(2.32) $\zeta_1 = \frac{p^2 - k^2}{2\Omega}\, \pi_1$

The solution for $\delta v_i$ is written from equation (2.6)

(2.33) $\delta v_i = C_3 \sinh kz$ ,

where the constant $C_3$ is determined from the conditions of continuity.

The corresponding solutions for $\pi_e$ , $u_z$ and $\zeta_e$ for the non-conducting fluid are

$$\pi_e = E_1 e^{-p'z} \quad , \quad z > h$$

(2.34) $\pi_e = {}^-E_1 e^{p'z} \quad , \quad z < h$

$$u_z = E_2 e^{m'z} - \frac{p' E_1 e^{-p'z}}{(p'^2 - m'^2)} \quad , \quad z > h$$

(2.35) $u_z = E_2 e^{m'z} - \dfrac{p' E_1 e^{p'z}}{(p'^2 - m'^2)} \quad , \quad z < h$

and

(2.36) $\zeta_e = \dfrac{p'^2 - k^2}{2\Omega} \pi_e$ ,

where

(2.37) $m'^2 = \dfrac{\sigma}{\nu_0} + k^2$ ,

(2.37a) $(p'^2 - k^2)(p'^2 - m'^2)^2 - \dfrac{4\Omega^2}{\nu_0^2} p'^2 = 0$ .

The solution for $\delta v_e$ is

$$\delta v_e = E_3 e^{-kz} \, , \, z > h \, ,$$

(2.38) $\delta v_e = - E_3 e^{+kz} \, , \, z < h$

The solution for $\bar{h}_e$ is written from (2.10), (2.11)

(2.39) $\bar{h}_e = \nabla (L e^{-kz}) \, , \, z > h,$

$$\bar{h}_e = \nabla (-L e^{-kz}) \, , \, z < h.$$

## 3. THE BOUNDARY CONDITIONS.

The perturbed interfaces between the conducting and non-conducting fluids considering the perturbations symmetrical about the mid-level of the layer are given by

(3.1) $z_1 = h + (\delta z) \exp. [ ( \sigma t + \overline{ik_1 x + k_2 y} ) ]$,

and

$z_2 = -h + (\delta z) \exp. [ ( \sigma t + \overline{ik_1 x + k_2 y} ) ]$,

where $(\Delta z)$ is the amplitude of the displacement at the interfaces.

The perturbation $\delta \hat{n}_0$ in the unit normal $n_0$ is given by

(3.2) $\delta \hat{n}_0 = (-ik_1 \Delta z, -ik_2 \Delta z, 0) \exp. (\sigma t + \overline{ik_1 x + k_2 y})$ .

The following conditions should be satisfied at the perturbed interfaces.

(i) gravitational potential is continuous i.e.

(3.3) $[ V ] = 0$

(ii) the normal component of the gradient of the gravitational potential is continuous, i.e.

(3.4) $\hat{n} . [ \nabla V ] = 0$ ,

(iii) the velocity should be compatible with the assumed form of the deformed interfaces

(3.5) $\hat{n} . [ \bar{u} ] = 0$ ,

(iv) the component of the magnetic field normal to the deformed interfaces must be continuous ,

(3.6) $\hat{n} . [ \bar{B} ] = 0$ ,

(v) all the perturbed quantities for the non-conducting medium must be bounded.

(vi) the tangential viscous stresses must be continuous

(3.7) $\left[ \left\{ \nu (D^2 + k^2) + \frac{\Omega^2 A}{\sigma} \right\} v_z \right] = 0$ ,

(vii) the normal component of the total stress tensor must also be continuous on the deformed interfaces

(3.8) $$[\, P \{\, \pi - \delta V - 2\, \nu D u_z \,\}\, ] = 0 \,,$$

(viii) integrating equation (2.24) over an infinitesimal element of $z$ at the interface, we get

(3.9) $$[\, \nu D v_z - \pi \,] = 0$$

where $\hat{n}$ denotes the unit normal vector directed in the conducting fluid, the square brackets denote the jump in the enclosed quantity upon crossing the interface from the non-conducting to the conducting fluid.

## 4. DISPERSION RELATION AND DISCUSSION.

The conditions (3.8) and (3.9) applied on equation (2.35) and (3.43) yield.

(4.1) $$\delta V_i = \frac{-4\pi G}{k\sigma} (\rho - \rho_0)\, e^{-kh} \,.\, (u_z)_{z=h} \sinh kz$$

and

$$\delta V_e = \frac{-4\pi G}{k\sigma} (\rho - \rho_0)\, e^{-kh}\, (u_z)_{z=h}\, e^{-kz} \,,\; z > h$$

(4.2) $$\delta V_e = \frac{+4\pi G}{k\sigma} (\rho - \rho_0) e^{-kh}\, (u_z)_{z=h}\, e^{kz} \,,\; z < h$$

The boundary condition (3.45) gives the perturbation in the magnetic field in the non-conducting fluid to be zero.

The boundary conditions (3.6), (3.7), (3.8) and (3.9) yield four homogeneous equations in four unknowns. The condition for the existence of non-trivial solutions yields the dispersion relation

(4.3) $$m \tanh X \Bigg[ \left\{ \nu(p^2+k^2) + \frac{\Omega^2 A}{\sigma} - \nu_0(p'^2+k^2) \right\} \left\{ \frac{\rho_0}{\rho p'} (m'-p') - \frac{2m'}{p'} \right\}$$

$$- (m'+p') \left\{ \frac{\nu}{p}(p^2-m^2) = \frac{F'}{\sigma k} - \frac{2m'^2 \nu_0}{p'} + \frac{\nu_0 \rho_0}{\rho p'} (p'^2+m'^2) \right\} \Bigg]$$

$$+ \frac{m^2}{p} \tanh y \Bigg[ \left\{ \nu(m^2+k^2) + \frac{\Omega^2 A}{\sigma} - \nu_0(p'^2+k^2) \right\} \left\{ \frac{m'}{p'} \left( \frac{p^2-m^2}{m^2} \right) \right.$$

$$-\frac{\rho_0}{\rho p'}(m'-p)\Big\}+(m'+p')\Big\{\frac{-F'}{k\sigma}+\frac{\nu_0 p_0}{\rho p'}\overline{p'^2+m'^2}-\frac{\nu_0 m'^2}{p' m^2}\overline{-p^2+m^2}\Big\}\Big]$$

$$+\frac{m'}{p'}(m^2-p^2)\left[\frac{\nu_0\rho_0}{\rho}(m'+p')-\frac{F}{\sigma k}\right]=0$$

where

$$X=mh,\ y=ph,\ \alpha=kh,$$

(4.4)

$$F'=4\pi G\rho\left(1-\frac{\rho_0}{\rho}\right)\alpha\left[1-\frac{1-\rho_0/\rho}{\alpha(1+\coth\alpha)}\right].$$

Making $\nu=\nu_0$ equation (4.3) reduces to

$$(4.5)\quad X\tanh x\left[\left\{(y^2-z^2)+\frac{L^2}{x^2-\alpha^2}\right\}\right]\alpha\left\{\frac{\rho_0}{\rho}\ \frac{x-z}{z}-\frac{2x}{z}\right\}-(x+z)$$

$$\left\{\frac{\alpha}{y}(y^2-X^2)-\frac{F}{x^2-\alpha^2}-\frac{2x^2\alpha}{z}+\frac{\rho_0\alpha}{\rho z}(z^2+x^2)\right\}\Big]+\frac{X^2}{y}\tanh y\left[\left\{(X^2-z^2)\right.\right.$$

$$\left.+\frac{L^2}{x^2-\alpha^2}\right\}\alpha\left\{\frac{x}{z}\ \frac{y^2+X^2}{X^2}-\frac{\rho_0}{\rho}\ \frac{x-y}{z}\right\}+(x+z)\left\{\frac{-F}{x^2-\alpha^2}+\frac{\rho_0}{\rho}\ \frac{\alpha}{z}(x^2+z^2)\right.$$

$$\left.\left.-\frac{\alpha}{z}\ \frac{x^2}{X^2}(y^2+X^2)\right\}\right]+\frac{x}{z}(X^2-y^2)\left[\frac{\rho_0}{\rho}\alpha(x+z)-\frac{F}{x^2-\alpha^2}\right]=0,$$

with $x=m'h$, $z=p'h$,

$$(4.6)\quad L^2=\frac{\Omega^2 Ah^4}{\nu^2},\ F=4\pi G\rho\left(1-\frac{\rho_0}{\rho}\right)\left(\frac{h^2}{\nu}\right)^2.\alpha\left[1-\frac{1-\rho_0/\rho}{(1+\coth\alpha)}\right],$$

and

$$(4.7)\quad X^2=x^2+\frac{L^2}{x^2-\alpha^2},\ \frac{\sigma h^2}{\nu}=x^2-\alpha^2\ .$$

The dispersion eq. (4.5) is discussed for certain limiting cases

(i) $k$ and $m$ small (ii) $k$ large (iii) $\nu\to\infty$ .

**Case (i) $k$ and $m$ small.**

The roots of the eq. (2.30*a*) neglecting $k^2 m^4$ are given by

(4.8) $p=\frac{1}{2}\left[(k^2+2m^2)\pm\left\{(k^2+2m^2)^2-4\left((2k^2m^2+m^4-\frac{4\Omega^2}{\nu^2}\right)\right\}^{1/2}\right]$

the dispersion eq. (4.5) reduces to

(4.9) $X^2=y^2$ i.e. $m^2=p^2$.

Eq. (4.9) with the help of eq. (4.8) gives

(4.10) $\sigma^2-\frac{4\ \Omega^2}{\nu^2k^2\ \sigma}+\Omega^2A=0$

of which both the roots are positive. Thus we find that long wave perturbations make the system unstable.

**Case** (ii) $k$ **large.**

From the eq. (2.30a) we find that for short wave perturbation, the effect of rotation is negligible. Making rotation evanescent, this case reduces to the problem of our paper[8] which has fully been discussed there. Thus for short waves, the unstable modes when the density $\rho_0$, of the surrounding medium is less than the density $\rho$, of the layer are shown in figures 1, 2, 3 and 4 for $\rho_0/\rho=0$ and 0.5. It is also seen that the system is physiclly unstable for $\rho_0/\rho>1$. In the limiting case of high viscosity the system exhibits marginal stabil'ty. It must be mentioned that the dispersion relation in this case is obtained independently taking $\Omega=0$.

**Case** (iii) $\nu=\nu_0=\infty$ (**The case of high viscosity**).

When viscosity is paramount, we put

(4.11) $X=\alpha+\delta$, $X=\alpha+\delta'$

where $\delta$ and $\delta'$ are small quantities and $y$ and $z$ tend to $\alpha$. Also equations (2.57) and (2.58) give

(4.12) $2\alpha\delta=\frac{\sigma h^2}{\nu}$, $2\alpha\delta'=\frac{\sigma h^2}{\nu}+\frac{L^2}{\sigma h^2/\nu}$

(4.13) $\frac{\sigma h^2}{\nu}=-\frac{2}{3}\frac{F}{\alpha^2}$

which implies that $\sigma$ is negative and first order. The perturbations are purely damped in this case. We also observe that rotation has no effect in this case.

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