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GENERALIZED HERMITE POLYNOMIALS

By

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1. INTRODUCTION.

Recently, Lahiri [3, 4, 5] studied the polynomials which are defined by the generating function*

$$(1.1)\quad e^{vxt-t^m} = \sum_{n=0}^{\infty} \frac{H_{n,m,v}(x)t^n}{n!},$$

where m is a positive integer.

In this paper; we shall extend some of the results due to Lahiri [4] with the help of the operator $x^2\frac{d}{dx}$ or by the modified operator $x^k\frac{d}{dx}$, studied in [1] and [2] respectively.

2. APPLICATION OF THE OPERATOR $\Omega = x^2 \frac{d}{dx}$.

For the operator $\Omega = x^2\frac{d}{dx}$, it is easy to show that, formally,

$$(2.1)\quad e^{t\Omega}\{f(x)\} = f\left(\frac{x}{1-xt}\right),$$

where $f(x)$ admits power series expansion.

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*More general sequences of polynomials have already been studied in the literature. See, for instance, Srivastava [6, 7].

Now consider the result [4, (3.1)]

$$\left(\frac{d}{dx}\right)^s \ [H_{n,m,v}(x)] = \frac{v^s \, n! \, H_{n-s,m,v}(x)}{(n-s)!} .$$

Replacing x by $1/x$, we get

$$(-\Omega)^s \ H_{n,m,v}(1/x) = \frac{v^s \, n! \, H_{n-s,m,v}(1/x)}{(n-s)!} ,$$

from which we can further write

$$e^{t\Omega} \{H_{n,m,v}(1/x)\} = \sum_{s=0}^{\infty} \frac{v^s \, n! \, (-t)^s}{(n-s)! \, s!} H_{n-s,m,v}(1/x).$$

Applying (2.1) and replacing x by $\frac{1}{x}$, we obtain the summation formula

$$(2.2) \quad H_{n,m,v}(x+t) = \sum_{s=0}^{n} \binom{n}{s} v^s \, t^s \, H_{n-s,m,v}(x) .$$

Now replacing t by x and x by 0, we obtain another result

$$(2.3) \quad H_{n,m,v}(x) = \sum_{s=0}^{n} \binom{n}{s} v^s \, x^s \, H_{n-s,m,v}(0).$$

In particular, on replacing t by $\frac{t}{v}$ in (2.2), we get

$$(2.4) \quad H_{n,m,v}\left(x + \frac{t}{v}\right) = \sum_{s=0}^{n} \binom{n}{s} t^s \, H_{n-s,m,v}(x) .$$

Similarly, on replacing x by $\frac{x}{v}$ in (2.3), we derive

$$(2.5) \quad H_{n,m,v}\left(\frac{x}{v}\right) = \sum_{s=0}^{n} \binom{n}{s} x^s \, H_{n-s,m,v}(0)$$

3. APPLICATION OF THE MODIFIED OPERATOR $x^k \frac{d}{dx}$.

Recently, using the operator $x^k \frac{d}{dx}$ in our previous paper [2], we studied the polynomials $T_n^{\alpha, k}(x, r, p)$ defined by the Rodrigues' formula*

$$T_n^{\alpha, k}(x, r, p) = x^{-\alpha} e^{px^r} \left(x^k \frac{d}{dx}\right)^n \left\{ x^{\alpha} e^{-px^r} \right\}.$$

Here k is not necessarily a positive integer; it may be any real number.

For the operator $\Omega_x = x^{\alpha} \frac{d}{dx}$, we can easily prove

$$(3.1) \quad e^{t\Omega_x} \{ f(x) \} = f\left[\frac{x}{\left(1-(\alpha-1)x^{\alpha-1}t\right)^{1/\alpha-1}}\right],$$

where α is any real number except 1, and $f(x)$ admits power series expansion.

Now consider the corrected version of formula (9.1 of Lahiri [4] :

$$(3.2) \quad \left(x^{\alpha m+1} \frac{d}{dx}\right)^r \left[x^{-\alpha n} H_{n, m, v}(x^{\alpha})\right] = \frac{n!(\alpha m)^r}{(n-mr)!} x^{-\alpha(n-mr)} H_{n-mr, m, v}(x^{\alpha}).$$

From the above result, we write

$$\sum_{r=0}^{\infty} \frac{n!(\alpha m)^r}{(n-mr)!} \frac{t^r}{r!} x^{-\alpha(n-mr)} H_{n-mr, m, v}(x^{\alpha})$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \left(x^{\alpha m+1} \frac{d}{dx}\right)^r \left\{x^{-\alpha n} H_{n, m, v}(x^{\alpha})\right\}$$

$$= e^{t\left(x^{\alpha m+1} \frac{d}{dx}\right)} \left\{x^{-\alpha n} H_{n, m, v}(x^{\alpha})\right\}$$

* A similar polynomial system was studied earlier by Srivastava and Singhal [8].

$$= \left[\frac{x}{(1-am\, x^{am}t)^{1/am}} \right]^{-an} H_{n,m,v} \left[\frac{x}{(1-am\, x^{am}t)^{1/am}} \right]^{a}.$$

Now replacing t by $\dfrac{t}{amx^{am}}$, and finally x^{a} by x, we get

$$(3.3) \qquad (1-t)^{n/m} H_{n,m,v}\left(\frac{x}{(1-t)^{1/m}}\right) = \sum_{r=0}^{[n/m]} \frac{t^r\, n!}{r!\,(n-mr)!} H_{n-mr,m,v}(x),$$

which is believed to be new.

Applying the same techniques, the above result can also be obtained from [4, (8.2)]

$$(3.4) \qquad \left(\frac{x^{m+1}}{m}\frac{d}{dx}\right)^{r} \left[\frac{x^{-n} H_{n,m,v}(x)}{n!} \right] = \frac{x^{-n+rm}}{(n-rm)!} H_{n-rm,m,v}(x).$$

Now we recall our earlier result [2]:

$$D^n = \sum_{j=0}^{n} \binom{n}{j} T_{n-j}^{a,k}(x,r,p)\, \Omega_x^{j},$$

where $D = ax^{k-1} - prx^{k+r-1} + \Omega_x$ and $\Omega_x = x^k \dfrac{d}{dx}$.

Replace k by $m+1$ in the above result, and denote

$$ax^{m} - prx^{m+r} + x^{m+1}\frac{d}{dx} \text{ by } \phi \text{ and } x^{m+1}\frac{d}{dx} \text{ by } \theta.$$

We thus obtain

$$\phi^{s} = \sum_{j=0}^{s} \binom{s}{j} T_{s-j}^{a,m+1}(x,r,p)\, \theta^{j},$$

whence we have

$$\phi^{s}\left\{\frac{x^{-n} H_{n, m, v}(x)}{n!}\right\}=\sum_{j=0}^{s}\binom{s}{j} T_{s-j}^{\alpha, m+1}(x, r, p) \theta^{j}\left\{\frac{x^{-n} H_{n, m, v}(x)}{n!}\right\}.$$

Now applying (3.4) on the right-hand side of the above result,

we get

$$(3.5)\quad \phi^{s}\left\{\frac{x^{-n} H_{n, m, v}(x)}{n!}\right\}$$

$$\sum_{j=0}^{min\,(s,[n/m])}\binom{s}{j}\frac{x^{-n+jm}\, m^{j}}{(n-jm)!} H_{n-jm, m, v}(x)\; T_{s-j}^{\alpha, m+1}(x, r, p),$$

which appears to be new.

In our previous paper [2], we also have a result

$$\Omega_{x}^{n}=\sum_{j=0}^{n}\binom{n}{j} T_{n-j}^{-\alpha, k}(x, r,-p)\; D^{j}.$$

Replying k by $m+1$, we write

$$\theta^{n}\left\{\frac{x^{-s} H_{s, m, v}(x)}{s!}\right\}=\sum_{j=0}^{n}\binom{n}{j} T_{n-j}^{-\alpha, m+1}(x, r,-p)\, \phi^{j}\left\{\frac{x^{-s} H_{, m, v}(x)}{s!}\right\}.$$

Now using (3.4), we establish the result

$$(3.6)\quad \frac{m^{n}\, x^{-s+nm}}{(s-nm)!} H_{s-nm, m, v}(x)$$

$$=\sum_{j=0}^{n}\binom{n}{j} T_{n-j}^{-\alpha, m+1}(x, r,-p)\, \phi^{j}\left\{\frac{x^{-s} H_{s, m, v}(x)}{s!}\right\}$$

We can easily prove

$$(3.7) \quad \theta^{s}(u.v) = \sum_{j=0}^{s} \binom{s}{j} \theta^{s-j}(u) \; \theta^{j}(v).$$

Thus we have

$$\theta^{s}\left\{\frac{x^{-n} H_{n, m, v}(x)}{n!} \cdot \frac{x^{-p} H_{p, m, v}(x)}{p!}\right\}$$

$$= \sum_{j=0}^{s} \binom{s}{j} \theta^{s-j}\left(\frac{x^{-n} H_{n, m, v}(x)}{n!}\right) \theta^{j}\left(\frac{x^{-p} H_{p, m, v}(x)}{p!}\right)$$

$$= \sum_{j=0}^{s} \binom{s}{j} \frac{m^{s-j} x^{-n+(s-j)m}}{(n-(s-j)m)!} H_{n-(s-j)m, m, v}(x)$$

$$\times \frac{m^{j} x^{-p+jm}}{(p-jm)!} H_{p-jm, m, v}(x) .$$

Hence we obtain

$$(3.8) \quad \theta^{s}\left(\frac{x^{-n-p}}{n!\,p!} H_{n, m, v}(x) \; H_{p, m, v}(x)\right)$$

$$= \sum_{j=0}^{s} \binom{s}{j} \frac{m^{s} x^{-n-p+sm}}{(n-(s-j)m)!\,(p-jm)!} H_{n-(s-j)m, m, v}(x) \; H_{p-jm, m, v}(x)$$

Remark:—Particularly for $m=v=2$, all the results of this section will be reduced to similar results for Hermite polynomials.

4. EXTENSION OF THE GENERATING FUNCTION.

From the generating relation (1.1), we write

$$\sum_{n=0}^{\infty} H_{n, m, v}(x+y) \frac{t^{n}}{n!} = e^{v(x+y)-t^{m}}$$

$$= e^{vty} \sum_{k=0}^{\infty} H_{k, m, v}(x) \frac{t^k}{k!}$$

$$= \sum_{n=0}^{\infty} \frac{(vyt)^n}{n!} \sum_{k=0}^{\infty} H_{k, m, v}(x) \frac{t^k}{k!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(vy)^{n-k}}{(n-k)!\, k!} H_{k, m, v}(x) t^n \quad .$$

Hence

$$(4.1) \quad H_{n, m, v}(x+y) = \sum_{k=0}^{n} \binom{n}{k} (vy)^{n-k} H_{k, m, v}(x) \quad .$$

Also

$$(4.2) \quad H_{n, m, v}\left(x+\frac{y}{v} \right) = \sum_{k=0}^{n} \binom{n}{k} y^{n-k} H_{k, m, v}(x) \quad .$$

Put $x=0$ in the above result, so that

$$(4.3) \quad H_{n, m, v}\left(\frac{y}{v}\right) = \sum_{k=0}^{n} \binom{n}{k} y^{n-k} H_{k, m, v}(0) \quad .$$

Again from (1.1), we have

$$\sum_{n=0}^{\infty} H_{n, m, (u+v)}(x) \frac{t^n}{n!} = e^{(u+v)\, xt - t^m}$$

$$= e^{uxt} \sum_{k=0}^{\infty} H_{k, m, v}(x) \frac{t^k}{k!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(ux)^{n-k}}{(n-k)!\, k!} H_{k, m, v}(x)\, t^n \quad .$$

Therefore,

$$(4.4) \quad H_{n, m, u+v}(x) = \sum_{k=0}^{n} \binom{n}{k} (ux)^{n-k} H_{k, m, v}(x) \quad .$$

Similarly

$$(4.5) \quad H_{n, m, u+v}(x) = \sum_{k=0}^{n} \binom{n}{k} (vx)^{n-k} H_{k, m, u}(x) \quad .$$

Replacing u by u/x in (4.4) and v by v/x in (4.5), we derive

$$(4.6) \quad H_{n, m, \frac{u}{x}+v}(x) = \sum_{k=0}^{n} \binom{n}{k} u^{n-k} H_{k, m, v}(x) \quad ,$$

and

$$(4.7) \quad H_{n, m, u+\frac{v}{x}}(x) = \sum_{=0}^{n} \binom{n}{k} v^{n-k} H_{k, m, u}(x) \quad .$$

Putting $u=0$ in (4.7), we further derive

$$(4.8) \quad H_{n, m, \frac{v}{x}}(x) = \sum_{k=0}^{n} \binom{n}{k} v^{n-k} H_{k, m, 0}(x) \quad .$$

Now comparing (4.2) and (4.6), we get

$$(4.9) \quad H_{n, m, \frac{u}{x}+v}(x) = H_{n, m, v}\left[x+\frac{u}{v} \right] \quad .$$

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