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P. HUMBERT'S CONFLUENT HYPERGEOMETRIC FUNTION ϕ_1 (α, β; γ; x, y)

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ABSTRACT :

Some known results involving Laguerre polynomials and Humbert's function are generalized.

1— INTRODUCTION.

A few years ago, Abdul-Halim and Al-Salam [1] obtained the expansion

$$(1.1) \qquad \phi_1 (\alpha, \beta; \gamma; -u, -uv) = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{(\gamma)_r} u^r L_r^{(-\beta-r)} (v)$$

where, as usual, the Laguerre polynomials are defined by

([6], p. 200)

$$(1.2) \qquad L_n^{(\alpha)} (x) = \frac{(1+\alpha)_n}{n!} {}_1F_1 \left[\begin{array}{c|c} -n & x \\ 1+\alpha & \end{array} \right].$$

${}_1F_1$ is the confluent hypergeometric function and ϕ_1 $(\alpha, \beta; \gamma; x, y)$ is the confluent form of Appell's function F_1 ([4], Vol II, p. 444)

$$(1.3) \qquad \phi_1 (\alpha, \beta; \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}$$

We prove here the more general result

$$1.4) \quad \phi_1 (\alpha, \beta; \gamma; -u, -u \sum_{i=1}^{n} w_i) = \sum_{r_1,\ldots,r_n=0}^{\infty} \frac{(\alpha)_{r_1+\ldots+r_n}}{(\gamma)_{r_1+\ldots+r_n}} u^{r_1+\ldots+r_n}$$

$$\times L_{r_1}^{(\beta_1-\beta-r_1)}(w_1) \ldots L_{r_{n-1}}^{(\beta_{n-1}-\beta_{n-2}-r_{n-1})}(w_{n-1}) L_{r_n}^{(-\beta_{n-1}-r_n)}(w_n)$$

with $(\beta_r - \beta_{r-1}) \neq 0, -1, -2, \ldots$ for $r = 1, 2, \ldots, n$ where $\beta_0 = \beta$, $\beta_n = 0$ and $|x| < 1$, $(\gamma) \neq 0, -1, -2, \ldots$.

We also obtain generalizations of known formulas which may be new, e. g. (2.1), (2.5), (3.2), (3.4), (3.7), (3.8), (3.9) and (3.12). Note that for the sake of brevity in most of these formulas, we do not mention explicitly the restrictions on the parameters involved. They can easily be obtained from the restrictions mentioned in connection with formulas (1.4) or (2.4). We refer the reader to standard texts, e. g. [5], for definitions and notation used in connection with hypergeometric functions.

2. PROOF OF THE MAIN RESULT.

Since it is no more difficult, we shall obtain slightly more general result than (1.4). Consider the identity

$$(2.1) \quad \phi_1 (\alpha, \beta; \gamma; x, y) = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{(\gamma)_r} (-x)^r L_r^{(\beta_1-\beta-r)} \left\{(1-\sigma)\frac{y}{x}\right\}$$

$$\phi_1 (\alpha+r, \beta_1; \gamma+r; x, \sigma y),$$

which is easily established as follow. From (1.2) and the fact that

$$(-r)_k = \frac{(-1)_k \, r!}{(r-k)!} \quad (0 \leq k \leq r) \text{ and}$$

$$\frac{(1+\beta_1-\beta-r)_r}{(1+\beta_1-\beta-r)_k} = (-1)^{r+k} (\beta-\beta_1)_{r-k},$$

the Laguerre polynomial of (2.1) can be written

(2.2)

$$L_r^{(\beta_1-\beta-r)}\left\{(1-\sigma)\frac{y}{x}\right\} = (-1)^r \sum_{k=0}^{r} \frac{(\beta-\beta_1)_{r-k}}{(r-k)!\,k!}\left((1-\sigma)\frac{y}{x}\right)^k.$$

With (2.2) and the definition (1.3), the right-side assumes the form

(2·3)

$$\sum_{r,\,m,\,n=0}^{\infty}\sum_{k=0}^{r}\frac{(a)_{r+m+n}}{(\gamma)_{r+m+n}}\frac{(\beta-\beta_1)_{r-k}(\beta_1)_m}{(r-k)!\,k!\,m!\,n!}x^{m+r-k}\,y^{n+k}\,\sigma^n\,(1-\sigma)^k$$

$$=\sum_{r,k,m,n=0}^{\infty}\frac{(a)_{r+m+n+k}}{(\gamma)_{r+m+n+k}}\frac{(\beta-\beta_1)_r\,(\beta_1)_m}{r!\,k!\,m!\,n!}x^{m+r}\,y^{n+k}\,\sigma^n\,(1-\sigma)^k$$

$$=\sum_{r,\,k=0}^{\infty}\sum_{m=0}^{r}\frac{(a)_{r+k}}{(\gamma)_{r+k}}\frac{(\beta-\beta_1)_{r-m}\,(\beta_1)_m}{(r-m)!\,(k-n)!\,m\,n!}x^r\,y^k\,\sigma^n\,(1-\sigma)^{k-n}$$

$$=\sum_{r,\,k=0}^{\infty}\frac{(a)_{r+k}(\beta-\beta_1)_r}{(\gamma)_{r+k}}\frac{x^r}{r!}\frac{((1-\sigma)\,y)^k}{k!}\,{}_1F_0\left(\begin{matrix}-k\\-\end{matrix}\,\middle|\,\frac{-\sigma}{1-\sigma}\right)\,{}_2F_1\left(\begin{matrix}-r,\ \beta_1\\1+\beta_1-\beta-r\end{matrix}\,\middle|\,1\right)$$

Gauss's summation theorem ([6], p. 49) implies (2·1) immediately.

Thus applying (2·1) n times we obtain our main result

(2.4)

$$\phi_1(a,\beta;\gamma;x,y)=\sum_{r_1,\ldots,r_n=0}^{\infty}\frac{(a)_{r_1+\ldots+r_n}}{(\gamma)_{r_1+\ldots+r_n}}(-x)^{r_1+\ldots+r_n}$$

$$\times L_{r_1}^{(\beta_1-\beta-r_1)}\left((1-\sigma_1)\frac{y}{x}\right)\ldots L_{r_n}^{(\beta_n-\beta_{n-1}-r_n)}\left((1-\sigma_n)\,\sigma_1\ldots\sigma_{n-1}\frac{y}{x}\right)$$

$$\times\,\phi_1(a+r_1+\ldots+r_n,\beta_n;\gamma+r_1+\ldots r_n;x,\sigma_1\ldots\sigma_n y),$$

which holds for $(\beta_i-\beta_{i-1})\neq 0,-1,-2,\ldots$ for $i=1,2,\ldots,n$

where $\beta_0=\beta$ and $|x|<1$, $(\gamma)\neq 0,-1,-2,\ldots\ldots$.

$$\text{If } x = -u, \quad y = -u \sum_{j=1}^{n+1} w_j \quad \text{and}$$

$$\sigma_i = 1 - \frac{W_i}{W_1 + \cdots + W_{n+1}} \quad \text{for } i = 1, 2, \ldots, n.$$

Eq (2.4) becomes

$$\phi_1\left(\alpha, \beta; \gamma; -u, -u \sum_{j=1}^{n+1} w_j\right) = \sum_{r_1, \ldots, r_n = 0}^{\infty} \frac{(\alpha)_{r_1 + \ldots + r_n}}{(\gamma)_{r_1 + \ldots + r_n}} u^{r_1 + \ldots + r_n}$$

$$(2{\cdot}5) \qquad \times L_{r_1}^{(\beta_1 - \beta - r_1)}(w_1) \ldots L_{r_n}^{(\beta_n - \beta_{n-1} - r_n)}(w_n)$$

$$\times \phi_1(\alpha + r_1 + \ldots + r_n, \; \beta_n; \; \gamma + r_1 + \ldots + r_n; \; -u, \; -u w_{n+1}).$$

Our relation (2·5) contains very many interesting special cases, listed in Sec. 3 of this paper. In particular we obtain (1·4) for $\beta_n = w_{n+1} = 0$.

3— SOME SPECIAL CASES.

If we recall that

$$(3.1) \qquad \phi_1(\alpha, \beta; \gamma; x, 0) = {}_2F_1\left(\left.\begin{matrix}\alpha, \beta \\ \gamma\end{matrix}\right| x\right),$$

then with $w_{n+1} = 0$ in (2·5), we obtain

(3·2)

$$\phi_1\left(\alpha, \beta; \gamma; -u, -u \sum_{j=1}^{n} w_j\right) = \sum_{r_1, \ldots, r_n = 0}^{\infty} \frac{(\alpha)_{r_1 + \ldots + r_n}}{(\gamma)_{r_1 + \ldots + r_n}} u^{r_1 + \ldots + r_n}$$

$$L_{r_1}^{(\beta_1 - \beta - r_1)}(w_1) \ldots L_{r_n}^{(\beta_n - \beta_{n-1} - r_n)}(w_n) \quad {}_2F_1\left(\left.\begin{matrix}\alpha + r_1 + \ldots + r_n, \beta_n \\ \gamma + r_1 \ldots + r_n\end{matrix}\right| -u\right);$$

as before we have two cases in which the ${}_2F_1$ can be summed. If $u = 1$ and $\gamma = 1 + \alpha - \beta_n$, then Kummer's summation theorem [(6), p. 68]

$$(3.3) \quad {}_2F_1\left(\left.\begin{matrix}\alpha, \beta \\ 1 + \alpha - \beta\end{matrix}\right| -1\right) = \frac{\Gamma(1 + \alpha - \beta)\,\Gamma(1 + \frac{1}{2}\alpha)}{\Gamma(1 + \frac{1}{2}\alpha - \beta)\,\Gamma(1 + \alpha)} \quad \text{Re}(\beta) < 1$$

and $1 + \alpha - \beta \neq 0, -1, -2, \ldots$
reduces (3.2) to

$$(3.4)\quad \phi_1\left(a, \beta; 1+a-\beta_n; -1, -\sum_{j=1}^{n} w_j\right)=\frac{\Gamma(1+a-\beta_n)}{\Gamma(a)}$$

$$\sum_{r_1,\ldots,r_n=0}^{\infty} \frac{\Gamma(1+\frac{1}{2}(a+r_1+\ldots+r_n))}{\Gamma(1-\beta_n+\frac{1}{2}(a+r_1+\ldots+r_n))\,(a+r_1+\ldots+r_n)}$$

$$\times\quad L_{r_1}^{(\beta_1-\beta-r_1)}(w_1)\ldots L_{r_n}^{(\beta_n-\beta_{n-1}-r_n)}(w_n)$$

with the supplementary conditions Re $(\beta_n) < 1$ and Re $(\beta) < 1$ for convergence.

In the second case, if $u=-1$ and Re $(\gamma-a-\beta_n) > 0$, we apply Gauss's summation theorem [(6), p. 49] to the right side of (3.2) and recalling that [(5), Vol I, p. 239]

$$(3.5)\quad \phi_1(a, \beta; \gamma; 1, y)=\frac{\Gamma(\gamma)\,\Gamma(\gamma-a-\beta)}{\Gamma(\gamma-a)\,\Gamma(\gamma-\beta)}\,{}_1F_1\left[\begin{matrix} a \\ \gamma-\beta \end{matrix}\middle|\, y\right] \text{ with}$$

Re $(\gamma-a-\beta) > 0$,

we obtain

$$(3{\cdot}6)\quad {}_1F_1\left(\begin{matrix} a \\ \gamma-\beta \end{matrix}\middle|\, \sum_{j=1}^{n} w_j\right) = \frac{\Gamma(\gamma-\beta)\,\Gamma(\gamma-a-\beta_n)}{\Gamma(\gamma-\beta_n)\,\Gamma(\gamma-a-\beta)}$$

$$\sum_{r_1,\ldots,r_n=0}^{\infty} \frac{(a)_{r_1+\ldots+r_n}\,(1-)^{r_1+\ldots+r_n}}{(\gamma-\beta_n)_{r_1+\ldots+r_n}}$$

$$\times\, L_{r_1}^{(\beta_1-\beta-r_1)}(w_1)\ldots L_{r_n}^{(\beta_n-\beta_{n-1}-r_n)}(w_n).$$

However in the above, if $a = -s$ and if $\gamma-\beta-1 = a_1+\ldots+a_{n+1}$, $\gamma-\beta_n-1 = a_{n+1}$ and $\beta_i-\beta_{i-1} = a_i$ for $i = 1, 2, \ldots, n$ with $\beta_0 = \beta$, then *E*q. (3·6) may be written

$$(3.7)\quad L_s^{(a_1+\ldots+a_{n+1})}\left(\sum_{j=1}^{n} w_j\right) = \sum_{r_1+\ldots+r_n=0}^{s}\left(\begin{matrix} a_{n+1}+s \\ a_{n+1}+r_1+\ldots+r_n \end{matrix}\right)$$

$$\times\, L_{r_1}^{(a_1-r_1)}(w_1)\ldots L_{r_n}^{(a_n-r_n)}(w_n)$$

where as usual

$$\binom{\alpha}{\beta} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)\,\Gamma(1+\alpha-\beta)}$$

Moreover, if $w_i = 0$ for $i = 1, 2, \ldots, n$, all polynomials reduce immediately to binomial coefficients and we obtain the interesting result

(3·8)

$$\binom{a_1+\ldots+a_{n+1}+s}{s} = \sum_{r_1+\ldots+r_n=0}^{s} \binom{a_{n+1}+s}{a_{n+1}+r_1+\ \ +r_n}\binom{a_1}{r_1}\binom{a_n}{r_n}.$$

Let us now return to the main relation (2.5) and, as before, put $u=-1$, $\alpha=-s$, $\gamma-\beta-1=a_1+\ldots+a_{n+1}$ and $a_i=\beta_i-\beta_{i-1}$ for $i=1, 2, \ldots, n$ with $\beta 0=\beta$ but with $w_{n+1}\neq 0$. Then $\gamma-\beta_n-1=a_{n+1}$ and the equation becomes

(3.9)

$$L_s^{(a_1+\ldots+a_{n+1})}\left[\sum_{j=1}^{n+1} w_j\right] =$$

$$\sum_{r_1+\ldots+r_n=0}^{s} L_{s-r_1-\ldots-r_n}^{(a_{n+1}+r_1+\ldots+r_n)}(w_{n+1})\, L_{r_1}^{(a_1-r_1)}(w_1)\ldots L_{r_n}^{(a_n-r_n)}(w_n)$$

Equation (3.9) is more general than (3.7), the special case is recovered from (3.9) with $w_{n+1}=0$. In (3.7), (3.8) and (3.9) the sum is extended over all sets of non-negative integers $r_1, r_2, \ldots, r_{n-1}$, and r_n having sums eqal to 0, 1,..., $s-1$ and s.

Another special case can be obtained. If we let $\beta\rightarrow\beta_1$ in (2.1) and apply the transformation

(3.10) $$\phi_1(\alpha, \beta; \gamma; x, y) = (1-x)^{-\beta}\; e^{y}\; \phi_1\left(\gamma-\alpha, \beta; \gamma; \frac{x}{x-1}, -y\right),$$

which is a confluent case of one of the known transformations of the function F_1 (Appell's first type) ([5], Vol. *I*, *p*. 239), we obtain

$$(3.11)\quad \phi_1\left(\gamma-\alpha, \beta; \gamma; \frac{x}{x-1}, -y\right) = e^{y(\sigma-1)}$$

$$\sum_{r=0}^{\infty} \frac{(\alpha)_r}{(\gamma)_r} \frac{[(1-\sigma)y]^r}{r!} \phi_1\left(\gamma-\alpha, \beta; \gamma+r; \frac{x}{x-1}, -\sigma y\right).$$

In the above, letting $\alpha=\gamma-\delta$, $x = \frac{-v}{1-v}$, and $y = -w$,

we have

$$(3\cdot12)\quad \phi_1(\delta, \beta; \gamma; v, w) = e^{-v(\sigma-1)} \sum_{r=0}^{\infty} \frac{(\gamma-\delta)_r}{(\gamma)_r} \frac{((\sigma-1)w)^r}{r!} \phi_1(\delta, \beta; \gamma+r; v, \sigma w).$$

This inverse relation can be obtained similarly. Moreover if $\beta=0$ and $\sigma = v$, we obtain a known result for the confluent hypergeometric function ${}_1F_1$ ([7], eq. 2·3·13, p. 23).

Many other special cases for two and three variables can be obtained from the relation (3·11), which are already special cases of other general known results. For example the special cases for three variables can be obtained from the bilinear relation (3·1) and (3.2) in ([8], p. 70) in terms of Kampé de Fériet's double hypergeometric functions. The special cases for two variables can be obtained from the well known formulas of Brafman ([2], eq. 27, p. 947) and Chaundy ([3], eq. (25), p. 62) which are also special cases of the more general formulas (3.1) and (3.2) in ([8], p. 70).

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