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m–SUBPARACOMPACT SPACES

BY

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A space X is said to be subparacompact [3] if every open covering of X admits a σ -discrete closed refinement. Subparacompact spaces have been studied by Arhangel'skii [1], McAuley [17], Čoban [7], Burke ([3], [4], [5]), Burke and Stoltenberg [6], Singal and Jain ([23], [24]) and others. Burke [3] proved that subparacompactness of X is equivalent to each of the following three properties of X :

(i) Every open covering of X has a σ -locally finite closed refinement.

(ii) Every open covering of X has a σ -closure preserving closed refinment.

(iii) If C is an open covering of X, then there exists a sequence $\{V_n\}_{n=1}^{\infty}$ of open coverings of X such that for each point x of X there is a positive integer m (x) and a $U \in C$ such that st (x, V_m (x)) $\subset C$. (St(x, V_m (x) denotes the union of all those members of V_m (x) which contain the point x.)

In [13] and [23], countably subparacompact spaces are also studied. X is said to be countably subparacompact if every countable open covering of X admits a σ -discrete closed refinement. This definition is due to Hodel [13]. Further, it has been proved in [23] that countable subparacompactness of X is equivalent to each of the following properties of X :

(a) Every countable open covering of X has a σ -locally finite closed refinement.

(b) Every countable open covering of X has a σ - closure preserving closed refinement.

In the present paper, we generalize the concept of subparacompactness to m-subparacompactness, where m is an infinite cardinal. A space X will be called m-subparacompact if every open covering of X of cardinality $\leqslant m$ has a σ-discrete closed refinement. If $m = \aleph_0$, then m-subparacompact spaces are precisely the countably subparacompact spaces. For a space X, having an open base of cardinality $\leqslant m$, m subparacompactness is equivalent to subparacompactness. In section 1 of the present paper, some characterizations and relationship of m-subparacompactness with other covering properties are obtained. Section 2 deals with subsets of m-subparacompact spaces. In section 3, direct and inverse preservation of m-subparacompact spaces under certain types of mappings are studied, and in section 4, some sum theorems have been proved. Invertibility, simple extention and adjunction of m-subparacompact spaces are discussed in section 6.

1. Characterizations

Lemma 1.1 Let every open covering of X of cardinality $\leqslant m$ have a σ-closure preserving closed refinement. For each $n \in \mathcal{N}$ let $U(n)=\{U\alpha(n): \alpha \in \Omega\}$ be an open covering of X with $|\Omega| \leqslant m$ and $U\alpha(n+1) < U\alpha(n)$ for all $\alpha \in \Omega$. Then there is a sequence $\{v_n\}_{n=1}^{\infty}$ of closed coverings of X such that, for each $n \in \mathcal{N}$, $v_n = \bigcup_{m=1}^{\infty} v_m\ n)$ and the following conditions are satisfied:

(1). $v_m(n)=\{V\alpha,m(n) : \alpha \in \Omega\}$ and is closure preserving for each $m \in \mathcal{N}$.

(2) $V_{\alpha,m}(n) \subset U\alpha(n)$ for each $\alpha \in \Omega$, $m \in \mathcal{N}$.

(3) $V_{\alpha,m}(n) \subset V_{\alpha,m+1}(n)$ for each $\alpha \in \Omega$, $m \in \mathcal{N}$.

(4) $V_{\alpha,m}(n+1) \subset V_{\alpha,m}(n)$ for each $\alpha \in \Omega$, $m \in \mathcal{N}$.

Proof. Proof is on the same lines as the proof of Lemma 3.1 in [23 and is therefore omitted.

Using the above lemma, the following theorem can be proved in the same way as Theorem 3.1 in [23].

Theorem 1.1 The following properties of X are equivalent :

(a) Every open covering of X of cardinality $\leqslant m$ has a σ-discrete closed refinement.

(b) Every open covering of X of cardinality $\leqslant m$ has a σ-locally finite closed refinement.

(c) Every open covering of X of cardinality $\leqslant m$ has a σ -closure preserving closed refinement.

From the above theorem and Theorem 1.1 in [19] it follows that every m-paracompact normal space is m-subparacompact. Below we give an example (Example 1.1) to show that the converse is not true.

In the next theorem we obtain a condition which together with m-subparacompactness implies m-paracompactness. But before that, we give definitions of some of the concepts which have been recently introduced by Krajewski [15] and Smith and Krajewski [15].

A Space X is said to be **expandable** if corresponding to each locally finite family $\{F\alpha : \alpha \in \Omega\}$ of subsets of X there is a locally finite open family $\{G\alpha : \alpha \in \Omega\}$ of subsets of X such that $F\alpha \subset G\alpha$ for each $\alpha \in \Omega$. X is said to be **almost expandable** if for each locally finite family $\{F\alpha : \alpha \in \Omega\}$ of subsets of X there exists a point-finite open family $\{G\alpha : \alpha \in \Omega\}$ of subsets of X such that $F\alpha \subset G\alpha$ for each $\alpha \in \Omega$. X is **discretely expandable** if for every discrete family $\{F\alpha : \alpha \in \Omega\}$ of subsets of X there exists a locally finite family $\{G\alpha : \alpha \in \Omega\}$ of open subsets of X such that $F\alpha \subset G\alpha$ for each $\alpha \in \Omega$. X is called **almost-discretely expandable** if for every discrete family $\{F\alpha : \alpha \in \Omega\}$ of subsets of X there exists a point-finite open family $\{G\alpha : \alpha \in \Omega\}$ of subsets X such that $F\alpha \subset G\alpha$ for each $\alpha \in \Omega$.

For any cardinal number m, the cardinality dependent concepts of m—expandability, almost m—expandability, discrete m—expandability, almost discrete m—expandability can be difined in a natural manner.

Theorem 1.2 A space X is m—paracompact if X is m—expandable and m - subparacompact.

Proof. Let X be an m-expandable m-subparacompact space and let $u = \{U\alpha : \alpha \in \Omega\}$ be an open covering of X of cardinality $\leqslant m$. Since X is m-subparacompact, there exists a σ-locally finite closed refinement $v = \bigcup_{i=1}^{\infty} V_i$ of u. Without any loss of generality, we can take $V_i = \{V_{i,\alpha} : \alpha \in \Omega\}$. Since X

is m-expandable, therefore for each $i \in N$ there is a locally finite open family $w_i = \{W_{i,\alpha} : \alpha \in \Omega\}$ of subsets of X such that $V_{i,\alpha} \subset W_{i,\alpha}$. Also for each $V_{i,\alpha}$ there exists a $U_{i,\alpha} \in u$ such that $V_{i,\alpha} \subset U_{i,\alpha}$. Let $G_{i,\alpha} = U_{i,\alpha} \cap W_{i,\alpha}$, and let $C_i = \{G_{i,\alpha} : \alpha \in \Omega\}$. Then $C = \bigcup_{i=1}^{\infty} G_i$ is a σ-locally finite open refinement of u. For each $i \in N$, put $C_i = \cup \{G : G \in G_i\}$ Then $\{G_i : i \in N\}$ is a countable open covering of X. X is countably paracompact, since $\aleph_0$ expandability is equivalent to countable paracompactness [15, Theorem 2.5]. Let $\{ H_\beta : \beta \in \Delta \}$ be a locally finite open refinement of $\{G_i : i \in N.\}$ For $\beta \in \Delta$, let $i(\beta) \in N$ such that $H_\beta \subset G_{i(\beta)}$. Then $\{H_\beta \cap G : G \in G_{i(\beta)}, \beta \in \Delta\}$ is a locally finite open refinement of u. Hence X is m-paracompact.

Corollary 1 1 A normal space X is m-paracompact if and only if X is m-expandable and m-subparacompact.

Example 1.1 An $\aleph_1$-subparacompact normal space which is not $\aleph_1$-paracompact.

Let F denote the normal but not collectionwise normal space constructed by Bing [2, Example G] where the underlying space P has cardinality $\aleph_1$. In [15] Krajewski proved that F is a normal metacompact Hausdorff space which is the countable union of closed paracompact subspaces (and hence subparacompact) but is not $\aleph_1$-expandable. Since every $\aleph$—paracompact space is $\aleph_1$-expandable [15, Theorem 2.4], therefore F is not even $\aleph_1$-paracompact.

We now give two theorems which exhibit relationship of m-subparacompact spaces with m-metacompact spaces. A space X is called **m-metacompact** if every open covering of X of cardinality $\leqslant m$ admits a point-finite open refinement.

Theorem 1.3 Every m-metacompact space in which every closed set is a G_δ - set is m-subparacompact.

Proof. Follows easily from Theorem 1 in [13].

Theorem 1.4 Every almost discretely m-expandable m-subparacompact space is m-metacompact.

Proof. Let $u = \{u_\alpha : \alpha \in \Omega\}$ be an open covering of X of cardinality $\leqslant m$. Since X is m-subparacompact, there is a σ-discrete closed refinement

$v = \bigcup_{i=1}^{\infty} v_i$. Without any loss of generality we can take $v_i = \{ V(i, \alpha) : \alpha \in \Omega \}$. Since X is almost discretely m-expandable there exists for each i, a point-finite open collection $k_i = \{ k(i, \alpha) : \alpha \in \Omega \}$ such that $V(i, \alpha) \subset k(i, \alpha)$ for all $\alpha \in \Omega$, For each $i \in \mathcal{N}$ and for each $\alpha \in \Omega$ there exists a member $U\alpha_i$ of u such that $V(i, \alpha) < U\alpha_i$. Take $G(i, \alpha) = k(i, \alpha) \cap U\alpha_i$. Then $C = \bigcup_{i=1}^{\infty} C_i$, where $C_i = \{ G(i, \alpha) : \alpha \in \Omega \}$ is a σ-point-finite open refinement of u. For each $i \in \mathcal{N}$ let $G_i = \bigcup \{ G(i, \alpha) : \alpha \in \Omega \}$. Then $\{ G_i : i \in \mathcal{N} \}$ is a countable open covering of X. Since for every infinite cardinal m, an m-subparacompact space is countably metacompact [13] the covering $\{ G_i : i \in \mathcal{N} \}$ admits a point finite open refinement $H = \{ H_\beta : \beta \in \Delta \}$. For $\beta \in \Delta$, let $i(\beta) \in \mathcal{N}$ such that $H_\beta \subset G_{i(\beta)}$. Then $\{ H_\beta \cap G : G \in C_{i(\beta)} ; \beta \in \Delta \}$ is a point finite open refinement of u. Hence X is m-metacompact.

Since every discrete family of subsets of a countably compact space is finite, the following result is immediate.

Theorem 1.5 Every m-subparacompact countably compact space is m-compact. (A space X is said to be m-compact if every open covering of X of cardinality $\leqslant m$ has a finite subcovering).

2. Subsets and m—subparacompactness

It can be easily verified that every closed subset of an m—subparacompact space is m—subparacompact. But, as with some other classes of topological spaces such as paracompact and collectionwise normality, a more general result holds.

Definition 2.1 A subset A of a space is said to be an $\mathbf{F_\sigma}$ **- subset** if it is a union of countable number of closed sets,

Definition 2.2 A subset A of X is said to be a **generalized $\mathbf{F_\sigma}$-subset** if for each open set U containing A there is an F_σ-subset B such that $A \subset B \subset U$.

Theorem 2.1 Every F_σ-subset of an m—subparacompact space is m—subparacompact.

Proof. Let A be an F_σ subset of an m—subparacompact space X. Let $U = \{ u_\alpha : \alpha \in \Omega \}$, where $| \Omega | \leqslant m$ be a relatively open covering of A. There exists a collection U^* of open subsets of X such that $U^* = \{U\alpha = \overset{*}{U}\alpha \cap A : U\alpha \in U\}$ Also there is a countable family $\{A_i\}$ of closed subsets of X such that $A = \bigcup_{i=1}^{\infty} A_i$.

For each i, let $w_i = U^* \cup \{X\text{-}A_i\}$. Then w_i is an open covering of X of cardinality $\leqslant m$. Let $v_i = \bigcup_{j=1}^{\infty} U_{i,j}$ be a σ-locally finite closed refinement of w_i. Let $V_{i,j}$ be the collection of all those members of $U_{i,j}$ which intersect A_i. Each $V_{i,j}$ is locally finite with respect to X. $\bigcup_{j=i}^{\infty} V_{i,j}$ is as dosed covering of A_i such that each member of $V_{i,j}$ is containrd in some $U\alpha^*$. Let $\omega_{i,j} = \{B \cap A : B \in U_{i,j}\}$ and let $w = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \omega_{i,j}$. Then w is a σ-locally finite (in A) closed (in A refinement of U. Hence A is m-subparacompact.

Theerem 2.2 If A is a subset of X such that every open subset of X which contains an m-subparacompact set that contains A, then A is m-subparacompact.

Proof. Let $U = \{U\alpha : \alpha \in \Omega\}$ be an open (in A) covering of A of cardinality $\leqslant m$. For each $\alpha \in \Omega$, let $V\alpha$ be an open subset of X such that $U\alpha = A \cap V\alpha$. Then $\bigcup_{\alpha \in \Omega} V\alpha$ is an open set containing A. Thus, by hypothesis there exists an m—subparacompact subset B of X such that $A \subset B \subset \bigcup_{\alpha \in \Omega} V\alpha$. Now $\{V\alpha \cap B : \alpha \in \Omega\}$ is an open (in B) covering of B of cardinality $\leqslant m$. Let $w = \bigcup_{i=1}^{\infty} w_i$ be a σ-locally finite (in B) closed (in B) refinement of $\{V\alpha \cap B : \alpha \in \Omega\}$. If we let $w_i = \{W_{ij} : j \in \Delta_i\}$ and $W_{ij} = V_{ij} \cap B$ where V_{ij} is closed in X. then each $V_{ij} \cap A$ is closed in A and the family $v_i = \{V_{ij} \cap A : j \in \Delta_i\}$ is locally finite in A for each i. Also $\bigcup_{i=1}^{\infty} v_i$ covers A. Hence A is m-subparacompact.

Corollary 2.1 Every generalized F_σ-subset of an m-subparacompact space is m-subparacompact.

Proof. Follows directly from the definition of generalized F_σ-subset and the above theorem.

Theorem 2.3 If every open subset of a space X is m—subparacompact then every subset of X is m—subparacompact.

Proof. Let A be any subset of X. Let $U=\{U_\alpha : \alpha \in \Omega\}$ be an open (in A) covering of A such that $|\Omega| \leqslant m$. For each $\alpha \in \Omega$, let U_α^* be an open subset of X such that $U_\alpha = A \cap U_\alpha^*$. Then $U^* = \{U_\alpha^* : \alpha \in \Omega\}$ is an open covering of $G = \bigcup_{\alpha \in \Omega} U_\alpha^*$ of cardinality $\leqslant m$. Since G is m—subparacompact, there exists a σ-locally finite closed (in G) refinemet $v^* = \bigcup_{n=1}^{\infty} v_n^*$ of U^*. For each $n \in N$ let $v_n = \{V \cap A : V \in v_n^*\}$. Then $v = \bigcup_{n=1}^{\infty} v_n$ is a σ-locally finite (in A) closed (in A) refinement of U. Hence A is m–subparacompact.

3. Mappings and m-subparacompactness

Theorem 3.1 Every closed continuous image of an m-subparacompact space is m-subparacompact.

Proof. Let $f : X \to Y$ be a closed continuous mapping of an m-subparacompact space X onto Y. Let $U=\{U_\alpha : \alpha \in \Omega\}$ be an open covering of Y of cardinality $\leqslant m$ Then $f^{-1}(U) = \{f^{-1}(U_\alpha) : U_\alpha \in U\}$ is an open covering of X of cardinality $\leqslant m$. Thus there is a σ-closure preserving closed refinement $v = \bigcup_{n=1}^{\infty} v_n$ of $f^{-1}(U)$. Since f is a closed and continuous mapping, each $f(v_n)$ is a closure preserving closed collecttion. Hence $f(v) = \bigcup_{n=1}^{\infty} f v_n)$ is a σ-closure preserving closed refinement of U. Hence Y is m subparacompact.

Definition 3.1 A mapping $f : X \to Y$ is said to be a **perfect mapping** if f is closed, continuous and $f^{-1}(y)$ is compact for each $y \in Y$.

Theorem 3.2 If X is a normal space and f is a perfect mapping from X onto Y, then X is m-subparacompact if Y is so.

Proof. Let $U=\{U_\alpha : \alpha \in \Omega\}$ be an open covering of X of cardinality $\leqslant m$. For each $y \in Y$ we can find a finite subcollection $U(y)$ of U such that $f^{-1}(y) \subset u(y) = \cup\{u : u \in U(y)\}$. Let $V(y) = Y - f(X - u(y))$. Then $v = \{V(y) : y \in Y\}$ is an open covering of Y Clearly, v is of cardinality $\leqslant m$. Since Y is m–subparacompact, therefore v has a σ-discrete closed refinement $v^* = \bigcup_{n=1}^{\infty} v_n^*$. Then $f^{-1}(v^*) = \{f^{-1}(V^*) : V^* \in v^*\}$ is a σ-discrete refinement of $\{u(y) : y \in Y\}$. Given $V^* \in v^*$, let $y(V^*)$ be a fixed element of Y such that $f^{-1}(V^*) \subset u(y(V^*))$. Let $u\{y(V^*)\} = \{u_1, \cdot, \cdot, u_{k(V^*)}\}$, so that $u(y(V^*))$ is a finite open covering of the normal space $f^{-1}(V^*$, Thus there is an open covering $\{u_1^*, u_2^*, \cdot\, u^*_{k(V^*)}\}$ of $f^{-1}(V^*)$ such that $u_i^* \subset u_i$ for all $i = 1, 2, \ldots, k(V^*)$ For each $n \in N$, let $w_n^* = \{f^{-1}(V^*) \cap u_i^* : V^* \in V_n^*, u_i \in U(y(V^*))\}$. We shall prove that each w_n^* is locally finite in X. Let $x \in X$. Since $f^{-1}(v_n^*)$ is a discrete collection in X, there exists a neighbourhood N_x of x which intersects at the most one member of $f^{-1}(v_n^*)$. Since each element of $f^{-1}(v_n^*)$ intersects at the most finitely many members of w_n^* and each member of w_n^* is contained in some element of $f^{-1}(v_n^*)$, it follews that N_x will intersect only finitely many members of w_n^*. Thus every open cover of $X$of cardinality $\leqslant m$ has a σ-locally finite closed refinement. Hence X is m-subparacompact.

4. Sum Theorems :

Sum theorems give conditions under which the union of topological spaces of given type are of the same type. Various sum theorems have been given for the class of paracompact, regular, completely normal, metrizable, m-paracompact and normal, subparacompact and other classes of spaces. In this section we shall obtian some sum theorems for the class of m-subparacompact spaces. It is easy to note that a countable union of closed m-subparacompact spaces is m-subparacompact. Throughout in the section, p will denote a topological property which is closed hereditary and which satisfies the locally finite sum theorem, which states the following :

'If $\{F_\alpha : \alpha \in \Omega\}$ is a locally fiinite closed covering of X such that each F_α possesses the property p, then X possesses p.'

Hodel [13] proved that p satisfies the following sum theorems.

Theorem 4.1 Let X be a topological space and let U be a σ-locally finite open covering of X such that the closure of each element of U possesses the property p. Then X possesses p.

Theorem 4.2 Let X be a topological space and let v be a σ-locally finite elementary covering (see Definition 4.1 below) of X. Then X possesses the property p if each $V \in v$ possesses p.

Theorem 4.3 Let X be a regular space and let v be a σ-locally finite open covering of X, each element of which possesses p and has the compact frontier. Then X has the property p.

Definition 4.1 A subset A of X is said to be **elementary** if it is open and is the union of a countable family of open subsets, the closure of each member of which is contained in A. A covering of X consisting of elementary sets is called an **elementary covering,**

Definition 4.2 (Y. Katuta, [14]). A family $\{A_\alpha : \alpha \in \Omega\}$ is said to be **order locally finite** if there is a linear ordering '$<$' on Ω such that for each $\alpha \in \Omega$, the family $\{ A_\lambda : \lambda < \alpha \}$ is locally finite at each point of A_α.

Every σ-locally finite family is order locally finite but not conversely .

In [22] Singal and Arya obtained the following sum theorems for p. These theorems generalise Theorems 4.1 and 4.3.

Theorem 4.4 Let v be an order locally finite open covering of a space X such that closure of each member of v possesses the property p. Then X possesses p.

Theorem 4.5 If X is a regular space, v is an order locally finite open covering of X each member of which possesses the property p and if the frontier of each member of v is compact, then X possesses the property p.

we shall prove that all these sum theorems hold for m-subparacompact spaces also. For that we have to first prove that the locally finite sum theorem holds for m-subparacompact spaces.

Theorem 4.6 If $\{F_\alpha : \alpha \in \Omega \}$ be a locally finite closed covering of X such that F_α is m-subparacompact, then X is m-subparacompact.

proof Let $u\{U_\beta : \beta \in \triangle \}$ be an open covering of X of cardlnality $\leq m$. For each $\alpha \in \Omega$, $\{U_\beta \cap F_\alpha : \beta \in \triangle\}$ is then an open (in F_α) covering of F_α of cardinality $\leq m$. Thus there exists a family $U^\alpha = \bigcup_{i=1}^{\infty} V_i^\alpha$ of closed subsets (of F_α and hence of X) such that each V_i^α is a discrete (in F_α and hence in X) family of subsets of F_α such that V^α is a covering of F_α. For each i, let $w_i = \bigcup_{\alpha \in \Omega} V_i^\alpha$ and let $w = \bigcup_{i=1}^{\infty} w_i$. Then w is a closed covering of X which is a refinement of U. Also it can be proved that each w_i is locally finite. Thus w is a σ - locally finite closed refinement of U and hence X is m-subparacompact.

Theorem 4.1 Every disjoint topological sum of m-subparacompact spaces is m-subparacompact.

In view of the Theorem 4.6 and the fact that m-subparacompactness is a closed hereditary property, we have the following results.

Theorem 4.7 If v is an order locally finite open covering of a space X such that the closure of each member of v is m-subparacompact, then X is m-subparacompact.

Theorem 4.8 If X is regular and V is an order locally finite open covering of X such that each member of v is m-subparacompact and the frontier of each member of v is compact, then X is m-subparacompact.

Theorem 4.9 If v is a σ -locally finite elementary covering of X such that each element of v is m-subparacompact, then X is m-subparacompact.

We now obtain some other sum theorems as consequences of the locally finite sum theorem for m-subparacompact spaces. The proofs of all these theorems follow same arguments as the corresponding theorems for subparacompact spaces in [23] and [24], We, therefore, only state these theorems.

Theorem 4.10 Let X be a regular space and let v be a locally finite open covering of X such that for each $V \in v$, V is m-subparacompact and $Fr\ V$ is Lindelöf. Then X is m-subparacompact.

Theorem 4.11 Let $U = \{U_\alpha : \alpha \in \Omega\}$ be a locally finite open covering of a normal space X such that each U_α is m-subparacompact space. Then X is m-subparacompact.

Theorem 4.12 If $\{U_\alpha : \alpha \in \Omega\}$ is a point finite open covering of a collectionwise normal space such that each U_α is m-subparacompact, then X is m-subparacompact.

An open covering U of X is said to be **noamai** if there exists a sequence $\{U_n : n=1,2,.....\}$ of open coverings or X such that each U_{n+1} is a star refinement of U_n (that is, the covering $\{St\ (x, U_{n+1}) : x \in X\}$ refines U_n) and U_1 is a refinement of U.

Theorem 4.13 Let $\{U\alpha : \alpha \in \Omega\}$ be a normal open covering of a normal space X such that each $U\alpha$ is m-subparacompact. Then X is m-subparacompact.

Therem 4.14 Let $\{U\alpha : \alpha \in \Omega\}$ be a σ-locally finite open covering of a normal space X such that $U\alpha$ is an F_σ -subset of X. Then X is m-subparacompact if each $U\alpha$ is m-subparacompact.

Theorem 4.15 Let $\{U\alpha : \alpha \in \Omega\}$ be a σ-locally finite open covering of a countably paracompact normal space X such that each $U\alpha$ is m- subparacompact. Then X is m-subparacompact.

Definition 4 3 [Frolik, 11]. A space X is said to be **weakly regular** if every open subset of X contains a non-empty regularly closed set.

Theorem 4.16 Every space which contains a proper non-empty regularly closed set is m–subparacompact if and only if every regularly closed subset of X is m-subparacompact.

Corollary 4.2 A weakly regular space X is m-subparacompact if and only if every proper regularly closed subset of X is m-subparacompact.

Corollary 4.3 A semi-regular space X is m-subparacompact if and only if every proper regularly closed subset of X, is m-subparacompact.

5. Embedding of m-subparacompact spaces

In [21] Mrowka proved that a locally m-paracompact completely regular space can be embedded in an m-paracompact space as an open subspace. We wish to prove the same for m-subparacompact spaces.

We need the following lemma.

Lemma 5.1 Let X be a regular space and let $\mathcal{G}$ be an open basis of neighbourhoods of a point $x \in X$ such that $X—G$ is m-subparacompact for each $G \in \mathcal{G}$, then X is m-subparacompact.

Proof. Let $U=\{U_\alpha : \alpha \in \Omega\}$ be an open covering of X of cardinality $\leqslant m$. For $x \in X$ there is a $\alpha_x \in \Omega$ such that $x \in U_{\alpha_x}$. Let $G_x \in \mathcal{G}$ be such that $x \in G_x \subset \overline{G_x} \subset U_{\alpha_x}$. Since $X - G_x$ is m—subparacompact and $\{(X-G_x) \cap U_\alpha : \alpha \in \Omega\}$ is an open (in $X-G_x$) covering of $X-G_x$ of cardinality $\leqslant m$, therefore there exists a σ-locally finite (in $X-G_x$ and hence in X) closed (in $X-G_x$ and hence in X) refinement $v=\bigcup_{n=1}^{\infty} v_n$ of $\{(X-G_x) \cap U_\alpha : \alpha \in \Omega\}$.

Let $v_0=\{\overline{G_x}\}$. Then $\bigcup_{n=0}^{\infty} v_n$ is a σ — locally finite closed refinement of U. Hence X is m—subparacompact.

Let us call a space locally m—**subparacompact** if each point of X has a neighbourhood whose closure is m—subparacompact.

Theorem 5.1 Every completely regular locally m-subparacompact space can be embedded in an m-subparacompact space as an open subspace.

Proof. Since m-subparacompactness is a closed hereditary property and is finitely additive for closed subsets, therefore Lemma 5.1 shows that m—subparacompactness satisfies the condition (w) of Mrowka [21]. Hence the result follows as in [21].

Corollary 5.1 Every completely regular locally subparacompact space can be embedded in a subparacompact space as an open subspace.

6. Invertibility, Simple Extensions and Adjunction of *m*-subparacompact Spaces.

Definition 6.1 [Doyle and Hocking, 9]. A space X is said to be **invertible** if for each open subset U of X there exists a homeomorphism $h : X \to X$ such that $h(X-U) \subset U$. h is called an **inverting homeomorphism** for U.

Theorem 6.1 Let X be an invertible space in one of its open sets U and let $\bar{u}$ be m-subparacompact. Then X is m-subparacompact.

Proof. Let $h : X \to X$ be an inverting homeomorphism for U. Then $h(X\text{-}U) \subset U$, and therefore $X=\bar{u} \cup h(\bar{u})$ where $\bar{u}$ and $h(\bar{u})$ are both m-subparacompact. Hence X is m-subparacompact, since a countable union of closed m-subparacompact subspaces is m-subparacompact.

Definition 6.2 [Levine, 16]. Let (X, T) be any topological space and let A be a subset of X such that $A \notin T$. Then the topology $T(A) = \{U \cup (V \cap A) : U, V \in T\}$ is called a **simple extension** of T.

It can easily be checked that $(A, T \cap A) = (A, T(A) \cap A)$ and $(X\text{-}A, T \cap (X\text{-}A)) = (X\text{-}A, T(A) \cap (X\text{-}A))$.

Theorem 6.2 If (X, T) is an m-subparacompact space and $A \subset X$ is such that $X\text{-}A \in T$, then $(X, T(A))$ is m-subparacompact if and only if $(X\text{-}A, T \cap (X\text{-}A))$ is m-subparacompact.

Proof. The 'only if' part follows from the facts that $(X\text{-}A, T \cap (X\text{-}A)) = (X\text{-}A, T(A) \cap (X\text{-}A))$ and every closed subspace of an m-subparacompact space is m-subparacompact. We shall now prove the 'if' part. Since (X, T) is m-subparacompact and A is a closed subspace of (X, T), it follows that $(A, T \cap A)$ is m-subparacompact. Thus if $(X\text{-}A, T \cap (X\text{-}A))$ is also m-subparacompact, X is the union of two closed m-subparacompact subspaces of $(X, T(A))$. Hence $(X, T(A))$ is m-subparacompact.

Theorem 6.3 If (X, T) is hereditarily m—subparacompact, and A is a subset of X such that $X\text{-}A \in T$, then $(X, T(A))$ is also hereditarily m—subparacompact.

Proof. Since (X, T) is hereditarily m-subparacompact, therefore $(X—A, T \cap (X—A))$ is also hereditarily m—subparacompact. The rest of the proof is on the same lines as the proof of Theorem 6.2.

Let X and Y be two topological spaces and let A be a closed subset of X. Let $f: A \to Y$ be a continuous mapping. Denote by $X+Y$ the disjoint topological sum of X and Y. Then an equivalence relation R may be defined as $(a, b) \in R$ if $a=b$ or $a=f(b)$ or $b=f(a)$. The quotient space $(X+Y)/R$ is called the adjunction space obtained by joining X to Y by means of the mapping f. The adjunction space is denoted by $X \underset{f}{\cup} Y$ and f is called the attaching map.

Let p denote the natural mapping (or projection) of $X+Y$ to $X \underset{f}{\cup} Y$; that is p maps a point of $X+Y$ to the equivalence class containing that point. We paracompact. Hence X is m-subparacompact, since a countable union of closed m-subparacompact subspaces is m-subparacompact.

Definition 6.2 [Levine, 16]. Let (X, T) be any topological space and let A be a subset of X such that $A \notin T$. Then the topology $T(A) = \{U \cup (V \cap A) : U, V \in T\}$ is called a **simple extension** of T.

It can easily be checked that $(A, T \cap A) = (A, T(A) \cap A)$ and $(X\text{-}A, T \cap (X\text{-}A)) = (X\text{-}A, T(A) \cap (X\text{-}A))$.

Theorem 6.2 If (X, T) is an m-subparacompact space and $A \subset X$ is such that $X\text{-}A \in T$, then $(X, T(A))$ is m-subparacompact if and only if $(X\text{-}A, T \cap (X\text{-}A))$ is m-subparacompact.

Proof. The 'only if' part follows from the facts that $(X\text{-}A, T \cap (X\text{-}A)) = (X\text{-}A, T(A) \cap (X\text{-}A))$ and every closed subspace of an m-subparacompact space is m-subparacompact. We shall now prove the 'if' part. Since (X, T) is m-subparacompact and A is a closed subspace of (X, T), it follows that $(A, T \cap A)$ is m-subparacompact. Thus if $(X\text{-}A, T \cap (X\text{-}A))$ is also m-subparacompact, X is the union of two closed m-subparacompact subspaces of $(X, T(A))$. Hence $(X, T(A))$ is m-subparacompact.

Theorem 6.3 If (X, T) is hereditarily m—subparacompact, and A is a subset of X such that $X\text{-}A \in T$, then $(X, T(A))$ is also hereditarily m—subparacompact.

Proof. Since (X, T) is hereditarily m-subparacompact, therefore $(X—A, T \cap (X—A))$ is also hereditarily m—subparacompact. The rest of the proof is on the same lines as the proof of Theorem 6.2.

Let X and Y be two topological spaces and let A be a closed subset of X. Let $f : A \rightarrow Y$ be a continuous mapping. Denote by $X+Y$ the disjoint topological sum of X and Y. Then an equivalence relation R may be defined as $(a, b) \in R$ if $a=b$ or $a=f(b)$ or $b=f(a)$. The quotient space $(X+Y)/R$ is called the adjunction space obtained by joining X to Y by means of the mapping f. The adjunction space is denoted by $X \underset{f}{\cup} Y$ and f is called the attaching map.

Let p denote the natural mapping (or projection) of $X+Y$ to $X \underset{f}{\cup} Y$; that is p maps a point of $X+Y$ to the equivalence class containing that point. We shall now show that the adjunction space $X \cup_f Y$ is m-subparacompact if X is m-subparacompact and Y is m-subparacompact and Hausdorff and A is a compact subspace of X.

We need the following result due to Dugundji [10; page 128]

Lemma 6.1 Let $X \cup_f Y$ be an adjunction space and let $p : X + Y \to X \cup_f Y$ be the projection mapping. If $F \subset X+Y$ is such that $F \cap X$ is closed in X, then $p(F)$ is closed if and only if $(F \cap Y) \cup f(F \cap A)$ is closed in Y.

Theorem 6.4 Let X be an m-subparacompact space, and Y be an m-subparacompact Hausdorff space. Then $X \cup_f Y$ is an m-subparacompact space if the domain of the attaching map f is compact.

Proof. If $A \subset X$ is compact and Y is Hausdorff then for a closed subset F of $X+Y$, $p(F)$ is closed since $F \cap Y$ and $f(F \cap A)$ are both closed. Thus the mapping $p : X+Y \to X \cup_f Y$ is a closed continuous mapping. Hence the result follows in view of Corollary 4.1 and Theorem 3.1.

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