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THERMAL STRESSES IN AN INFINITE SOLID CONTAINING A CYLINDRICAL CAVITY AND TWO STRIP CRACKS

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ABSTRACT

The thermoelastic equilibrium of an infinite solid containing a cylindrical cavity and two strip cracks situated symmetrically on a diametral plane of the cavity, is investigated. It is shown that the problem is equivalent to one finding distribution of thermal stress in an infinite two dimensional medium containing a circular hole and two Griffith cracks. By making suitable representations for the complex potentials occuring in the equatiuns of thermoelastic equilibrium, the problem is reduced to a set of triple integral equations with trigonometrical kernels. They are solved by using the technique of Finite Hilbert transform. Finally, the analytical expressions for quantities of physical interest are also found.

1. INTRODUCTION :

Recently the authors [1] have studied the Problem of determining the distribution of thermal stresses inside an infinite cylinder containing two strip cracks. It has been noticed that there is concentration of stresses and temperature at origin. The damages in the elastic material due to this phenomena can be avoided by creating a hole at the origin. The problem of a hole and two coplanar Griffith cracks is studied here. It is equivalent to the problem of determining the thermal stresses inside an infinite solid containing a cylindrical cavity and two strip cracks. The method of solution is a direct extension of the corresponding elastic problem [2].

2. FORMULATION OF PROBLEM :

We shall consider the temperature and displacement fields in an infinite elastic solid containing a cylindrical cavity and two strip cracks lying symmetrically on a diametral plane of cavity. The material of the solid is supposed to be homogeneous and isotropic with regard to both thermal & mechanical properties. The modulus of rigidity of the solid is denoted by μ and Poisson's ratio by η: The two dimensional problem is considered within the limits of the classical sheory of infinitesimal elasticity. If we use cylindrical polar coordinates, the cylindrical cavity Is supposed to occupy she region $r=\rho < a < b, 0 \leqslant \theta \leqslant 2\pi, -\infty < z < \infty$ with the axis of the cylinder being z — axis. The two strips cracks occupy the region $a < r < b, \theta = 0, \pi$ and $-\infty < z < \infty$. The state of stress in the solid is created by same temperature along the whole length of cracks so that the thermal stress field is the same in all the planes perpendicular to the axis of the cylinder. The above said problem is thus equivalent to one of finding the distribution of stress in an infinite two-dimensional medium containing a circular hole and two Griffith cracks situated symmetrically on a diameter of the hole. The displacement vector **U**, for symmetrical deformation, is taken to have components $(u_r, u_\theta, 0)$ and stress tensor $(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta})$ The boundary conditions to be satisfied on $\theta=0$ are

$$\sigma_{\theta\theta}(r,\theta) = 0 \qquad , a < r < b \tag{2.1}$$

$$\sigma_{r\theta}(r,\theta) = 0 \qquad , \rho < r < \infty \tag{2.2}$$

$$u_\theta(r,\theta) = 0 \qquad , r > b, \rho < r < a. \tag{2.3}$$

Since temperature on the surface of the crack is prescribed we have on $\theta=0$,

$$T(r,\theta) = T(r) \qquad , a < r < b \tag{2.4}$$

$$\frac{\partial T(r,\theta)}{r\,\partial\theta} = 0 \qquad , \rho < r < a, \quad r > b. \tag{2.5}$$

Further, the lateral surface of cylindrical cavity is supposed to be free of traction and is kept at zero temperature. Hence

$$\sigma_{rr}(\rho,\theta) = \sigma_{r\theta}(\rho,\theta) = T(\rho,\theta) = 0, \; 0 \leqslant \theta \leqslant \pi/2. \tag{2.6}$$

In addition, we have on $\theta = \pi/2$

$$\sigma_{r\theta}(r,\pi/2) = u_\theta(r,\pi/2) = T(r,\pi/2) = 0, \text{ for all } r. \tag{2.7}$$

3. THE EQUATIONS OF EQUILIBRIUM FOR THE THERMOELASTIC FIELD.

We quote from [1] the basic equations for two dimensional isotropic elasticity in presence of temperature field. If r,θ are polar coordinates and

$z = re^{i\theta}$, $\bar{z} = re^{-i\theta}$, then

$$\mu(u_r + iu_\theta) = \left[k\phi(z) - \overline{\phi(z)} - (z - \bar{z})\,\overline{\phi'(z)} - \overline{\omega(z)}\right]e^{-i\theta} + \left(v,_r + \frac{i}{r}v,_\theta\right) \tag{3.1}$$

$$\sigma_{rr} + \sigma_{\theta\theta} = 4\left[\phi'(z) + \overline{\phi'(z)}\right] - 2KT \tag{3.2}$$

$$\sigma_{rr} + i\sigma_{r\theta} = 2\left[\phi'(z) + \overline{\phi'(z)}\right] - 2\left[(z - \bar{z})\phi''(z) + \overline{\omega'(z)}\right]e^{-2i\theta} - KT + \left[\left(v,_{rr} - \frac{1}{r}v,_r - \frac{1}{r^2}v,_{\theta\theta}\right) - 2i\left(\frac{1}{r^2}v,_\theta - \frac{1}{r}v,_{r\theta}\right)\right] \tag{3.3}$$

$$\sigma_{\theta\theta} - i\sigma_{r\theta} = 2\left[\phi'(z) + \overline{\phi'(z)}\right] + 2\left[(z - \bar{z})\,\overline{\phi''(z)} + \overline{\omega'(z)}\right]e^{-2i\theta} - KT - \left[\left(v,_{rr} - \frac{1}{r}v,_r - \frac{1}{r^2}v,_{\theta\theta}\right) - 2i\left(\frac{1}{r^2}v,_\theta - \frac{1}{r}v,_{r\theta}\right)\right] \tag{3.4}$$

where μ is the modulus of rigidity, $k = (3 - 4\eta)$ for plane strain and $k = (3-\eta)/(1+\eta)$ for generalized plane stress; η being Poisson's ratio. $K = \frac{E_{at}}{1-\eta}$ for plane strain and $K = E_{at}$ for plane stress; E being Young's modulus and $_{at}$ the coefficient of linear expansion of the solid. $T(r,\theta)$ is the prescribed temperature field and $v(r,\theta)$ is a potential function satisfying the equation

$$\nabla^2 v = KT \tag{3.5}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$.

The temperature field in the solid is determined by the Laplace equation

$$\nabla^2 T = 0 \tag{3.6}$$

in the steady state and in the absence of heat source. The appropriate expression for T satisfying (3.6) in the region undcr consideration, i.e. an infinite medium with a circular hole, is given by

$$T(r,\theta) = \int_0^\infty \frac{A(\xi)}{\xi} e^{-\xi r \sin\theta} \cos(\xi r \cos\theta)\, d\xi + \sum_{n=o}^{\infty} a_n r^{-n} \cos n\theta. \tag{3.7}$$

With this choice for $T(r,\theta)$, the appropriate expressien for v satisfying (3.5) can be written as

$$v(r,\theta) = -\frac{K}{2}\int_0^\infty \frac{A(\xi)}{\xi^3}(1+\xi r \sin\theta)\, e^{-\xi r \sin\theta} \cos(\xi r \cos\theta)\, d\xi -$$

$$-\frac{K}{4}\sum_{n=o}^{\infty} \frac{a_n}{(n-1)} r^{-n+2} \cos n\theta. \tag{3.8}$$

The expressions for complex potentials $\phi(z)$ and $\omega(z)$ occuring in (3.1) to (3.4), we quote from the corresponding eleastic problems already solved by the first two authors [2]. These are

$$\phi(z) = \int_0^\infty i B(\xi)\, e^{i\xi z} d\xi + \sum_{n=o}^{\infty} \frac{C_n}{n+1} x^{-n-1} \tag{3.9}$$

$$\omega(z) = \sum_{n=o}^{\infty} \frac{d_n}{(n+1)} z^{-n-1}. \tag{3.10}$$

The expressions for components of nisplacement vector and stress tensor can be found by substituting the values $T(r,\theta)$, $v(r,\theta)$, $\phi(z)$, $\omega(z)$, and their corresponding derivatives in (3.1) to (3.4) and, then separating the real and imaginary parts.

THE TEMPERATURE FIELD :

The temperature field satisfies conditions (2.4) and (2.5) on $\theta=0$. These give rise to the following set of triple integral equations to desermine the unknown function $A(\xi)$ occuring in $T(r,\theta)$. The integral equations are :

$$\left.\begin{aligned} &\int_0^\infty A(\xi)\cos\xi r\, d\xi = 0 \quad , \rho < r < a \\ &\int_0^\infty \frac{A(\xi)}{\xi}\cos\xi r\, d\xi = \frac{\pi}{2} P(r), \quad a < r < b \\ &\int_0^\infty A(\xi)\cos\xi r\, d\xi = 0 \quad , r > b \end{aligned}\right\} \tag{4.1}$$

where

$$P(r) = \frac{2}{\pi}\left[T(r) - \sum_{n=0}^{\infty} a_n r^{-n} \right].$$

The solution for such a set of triple integral equations was found by Srivastava and Lowengrub [3] by utilizing the Finite Hilbert transform technique. Hence, quoting the result from [3] we obtain

$$A(\xi) = \int_a^b \frac{h(t^2)}{t}(1-\cos\xi t)\, dt \tag{4.2}$$

where the unknown function $h(t^2)$ is determined from the condition

$$h(t^2) = \mathrm{H}\left[rP'(r) \right] + C' / \left\{ (t^2-a^2)(b^2-t^2) \right\}^{\frac{1}{2}}. \tag{4.3}$$

Here C' is an arbitrary constant to be found from the condition

$$\int_a^b \frac{h(t^2)}{t}\, dt = \frac{2}{\log\left\{\frac{b-a}{b+a}\right\}} \int_a^b \frac{rP(r)\, dr}{\left\{(r^2-a^2)(b^2-a^2)\right\}^{\frac{1}{2}}}. \tag{4.4}$$

To complete the solution, we satisfy the boundary condition $T(\rho,\theta) = 0,\ 0 \leqslant \theta \leqslant \pi/2$. This gives us a relationship between the unknown function $h(t^2)$ and the coefficient a_n.

Thus, $T(\rho,\theta) = 0$ yields

$$-\int_a^b \left[\log(\rho/t) + \sum_{n=1}^{\infty} \frac{1}{2n} (\rho/t)^{2n} \cos 2n\theta \right] \frac{h(t^2)}{t}\, dt +$$

$$\sum_{n=0}^{\infty} a_{2n}\ \rho^{-2n} \cos 2n\theta = 0.$$

The above equation tells us that all odd coefficients a_{2n+1} are zero and even coefficients are given by

$$a_0 = \int_a^b \frac{h(t^2)}{t} \log(\rho/t)\, dt \tag{4.5}$$

and

$$a_{2n} = \int_a^b \frac{1}{2n} \left\{ \frac{\rho^2}{t} \right\}^{2n} \frac{h(t^2)}{t}\, dt,\ n \geqslant 1. \tag{4.6}$$

Substituting the value of the coefficient in the expression for $P(r)$, we find that

$$P(r) = \frac{2}{\pi} \left[T(r) + \int_a^b N(x,r)\ N(x,r) \frac{h(x^2)}{x}\, dx \right] \tag{4.7}$$

where

$$N(x,r) = -\log(\rho/x) - \sum_{n=1}^{\infty} \frac{1}{2n} \left(\frac{\rho^2}{rx} \right)^{2n}$$

or

$$N(x,r) = -\left[\sum_{n=1}^{\infty} \frac{(1-\rho/x)^n}{n} + \sum_{n=1}^{\infty} \frac{y^{2n}}{2n} \right].$$

$$M\ (x^2, t^2) = \frac{1}{\pi\ s\ (\ t^2\ )}\left[\ \frac{a}{x}\ \log\ (\ \rho/x\ ) + (\ \beta + \frac{2\ a\ b}{t^2}\ )\ \frac{\rho^4}{x^3}\ + 0\ (\ \rho^8\ )\ \right]$$

$$a = \frac{4\ a\ b}{\log \left\{ \frac{b-a}{b+a} \right\}},\ \beta = \frac{-1}{2\ a\ b}\left[\ a + 2\ (\ a^2 + b^2\ )\ \right].$$

The equation (4.10) is a Fredholm integral equation of the second kind. If we observe the kernel $M\ (x^2, t^2)$ carefully, we find that it has got weak sigularities at the end points. Inspite of this, all fundamental Fredholm theorems are applicable if they are stated in a suitable manner. This will be shown by the reduction of (4.10) to a Fredholm integral equation with a bounded $L^2\ (a, b)$ kernel. Writing (4.10) as

$$H\ (t^2) = \psi\ (t^2)\ S\ (t^2) + \int_a^b H\ (\ x^2\ )\ \frac{R\ (x^2, t^2)}{S\ (x^2)} \mathrm{dx} \tag{4.11}$$

where

$$H\ (t^2) = h\ (t^2)\ S\ (t^2),\ R\ (x^2, t^2) = M\ (x^2, t^2)\ S\ (t^2).$$

The first term on the right hand side of (4.11) is easily shown to be bounded. Further $R\ (x^2, t^2)$ is also bounded with respect to the variable t. The variable x will now be replaced by θ, by writing $x^2 = a^2\ \sin^2\ \theta + b^2\ \cos^2\ \theta$. With this transformation (4.11) reduces to

$$H\ (t^2) = \psi\ (t^2)\ S\ (t^2) + \int_0^{\pi/2} H\ (x^2)\ \frac{R\ (x^2, t^2)}{x^2}\ d\theta. \tag{4.12}$$

This is a Fredholm integral equation with bounded $L^2\ (0, \pi/2)$ kernel Thus a solution of (4.10) is equivalent to the solution of (4.12) in ordinary sense. Consequently we conclude that the Fredholm theorems are valid for (4.10).

We now solve the Fredholm integral equation (4.10) by the method of successive approximations. Let the first approvimation be

$$h_0\ (t^2) = \psi\ (t^2). \tag{4.13}$$

The second approximation is obtained by substitut ng (4·13) on the right sibe of (4·10) and so on. We shall now consider the physically important case when the crackes are opened by constant temperature i. e. T(r) = T. In this case, we obtain

$$h_0(t^2) = \frac{P}{S\ (t^2)},\quad P = \frac{\alpha\ T}{\pi} \tag{4.14}$$

$$h_1(t^2) = \psi(t^2) - \frac{P}{\pi S(t^2)} \int_a^b \frac{\alpha \log\ (\rho/x) + (\ \beta + \ 2ab/t^2)\ \rho^4/x^2 + O(\rho^8)}{x\ \{\ (x^2 - a^2)\ (b^2 - x^2)\ \}^{\frac{1}{2}}}\, dx$$

$$= \frac{\rho}{S\ (t^2)} \left[\alpha_1 + \alpha_2 \ \left(\beta + \frac{2ab}{t^2} \right) \ \rho^4 + O(\rho^8) \ \right] \tag{4.15}$$

where

$$\alpha_1 = 1 + \frac{\alpha}{2ab} \ \log\ \frac{\rho\ (a+b)}{2ab},\quad \alpha_2 = \frac{(a^2+b^2)}{4a^3\ b^3}.$$

The third approximation is

$$h_2\ (\ t^2\ ) = \frac{P}{S(\ t^2\ )} \left[\beta_1 + \left\{ \beta_2 + \alpha_1 \alpha_2 \left(\beta + \frac{2ab}{t^2} \right) \right\} \ \rho^4 + O\ (\rho^8) \ \right], \tag{4·16}$$

where $\beta_1 = 1 - \alpha_1 + \alpha_1{}^2$, $\beta_2 = \alpha_2 \left[(\alpha_1 - 1)\ \ (\beta + 4a^2b^2\alpha_2) + \alpha \left(2ab\ \alpha_2 - \frac{1}{2ab} \right) \right]$

The higher approximations can be obtained by continuing the process. In deriving (4·15) and (4·16) we have used results (12) and (13) of the appendix.

The expressions for the temperature field can be easily derived, they are

$$T\ (r,\ O) = \int_0^\infty \frac{A\ (\ \xi\ )}{\xi} \ \cos\ \xi r.\ d\xi + \sum_{n=0}^\infty \ a_n\ r^{-n} \ .$$

$$= \int_a^b \frac{\log\ |\ 1 - t^2/r^2\ |}{t} \ h\ (t^2)\ dt + \sum_{n=0}^\infty \ a_{2n}\ r^{-2n} \tag{4.17}$$

using results (14) and (15) of the appendix, we obtain

$$T(r,0)=\begin{cases} \frac{a\,T}{2\,a\,b}\Big[\,2\left\{\beta_1+\ (\beta_2+\alpha_1\alpha_2\beta)\rho^4\right\}\log\frac{a\sqrt{(b^2-r^2)}+b\sqrt{(a^2-r^2)}}{r\,(a+b)}+ \\ +\Big(2+\frac{2\,(R-a\,b)}{r^2}+\frac{a^2+b^2}{a\,b}\Big\{\log\Big(\frac{b-a}{b+a}\Big)+ \\ \log\frac{a^2(b^2-r^2)+b^2(a^2-r^2)+2\,a\,b\,R}{r^2\,(b^2-a^2)}\Big\}\Big)\alpha_1\,\alpha_2\,\rho^4+O\,(\rho^8)+Q\,(r),\Big] \\ \qquad\qquad \rho<r\leqslant a \\ T\,(r)=T, \qquad a\leqslant r\leqslant b \\ \frac{a\,T}{2\,a\,b}\Big[\,2\left\{\,\beta_1+(\beta_2+\alpha_1\alpha_2\beta)\rho^4\,\right\}\log\frac{a\sqrt{(r^2-b^2)}+b\sqrt{(r^2-b^2)}}{r\,(a+b)}+ \\ +\Big(\,2-\frac{2\,(R_1+a\,b}{r^2}+\frac{(a^2+b^2)}{a\,b}+\Big\{\log\Big(\frac{b-a}{b+a}\Big)+ \\ \log\frac{a^2\,(r^2-b^2)+(r^2-a^2)+2\,a\,b\,R_1}{r^2\,(b^2-a^2)}\Big\}\Big)\alpha_1\,\alpha_2\,\rho^4+ \\ +O\,(\rho)+Q\,(r),\Big] \qquad r>b \end{cases} \tag{4.18}$$

where

$$R=\{(a^2-r^2)\,(b^2-r^2)\}^{1/2},\ R_1=\{(r^2-a^2)\,(r^2-b^2)\}^{1/2}$$

and

$$Q\,(r)=a\,T\Big[\,d_0+(d_2-\frac{\alpha_2\,\beta_1)}{2\,r^2}\,\rho^4+O\,(\rho^8)\Big],$$

$$d_0=\frac{(\alpha_1-1)\,\beta_1}{a}\,,\ d_2=(\beta_2+\alpha_1\,\alpha_2\,\beta)\,\frac{d_0}{\beta_1}+\frac{\alpha_1\,\alpha_2}{a\,b}\Big\{(a^2+b^2)\,\frac{d_0}{\beta_1}+$$

$$\frac{(a-b)^2}{4\,a\,b}\Big\}.$$

5. The Thermoelastic field :

We divide the solution of the problem into two parts.

Condition on cracks Face : In this part we consider the relations which exist between arbitrary functions and the unknown coefficients C_n and d_n, if boundary conditions on the line $\theta=O$ and $\theta=\pi/2$ are to be satisfied. The boundary conditions (2·7) on $\theta=\pi/2$ are statisfied if and only if the coefficients C_{2n+1} and d_{2n+1} are all zero. With this choice of C_{2n+1} and d_{2n+1}, we see that (2.2) is also satisfied. The conditions (2.1) and (2.3) lead to the triple integral equations

$$\left.\begin{array}{ll}\int_0^\infty B(\xi)\cos\xi r\, d\xi = O, & O<r<a\\ \int_0^\infty \xi B(\xi)\cos\xi r\, d\xi = \frac{\pi}{2}G(r), & a<r<b\\ \int_0^\infty B(\xi)\cos\xi r\, d\xi = O, & r>b\end{array}\right\}$$

where

$$G(r)=-\frac{2}{\pi}\sum_{n=o}^{\infty}\left[(2C_n+d_n)r^{-n-2}+\frac{K}{2}(n+2)a_n r^{-n}+K\int_0^\infty\frac{A(\xi)}{\xi}\cos\xi r.\, d\xi\right]. \tag{5.2}$$

Also we have used the following definitions

$$B(\xi)=2B(\xi) \quad \text{and } A(\xi)=2A(\xi). \text{ in deriving} \tag{5.1}$$

The solution of the above set of triple integral equations as given by Srivastava and Lowengrub [3] is

$$B(\xi)=\frac{1}{\xi}\int_a^b g(t^2)\sin\xi t.\, dt \tag{5.3}$$

where $g(t^2)$ is determined by

$$g(t^2)=-H[G(r)]+C/\{(t^2-a^2)(b^2+t^2)\}^{1/2} \tag{5.4}$$

satisfying the condition $\int_a^b g(t^2)\,dt = O$ and C is an arbitrary constant.

Thus, (5.4) gives the relation connecting $g(t^2)$ and C_{2n} and d_{2n}.

Conditions on the free boundary : We now complete the solution by satisfying the boundary conditions (2.6).

Replacing $\beta(\xi)^*$ in (3.9) by (5.3) and noting that for $z = \rho e^{i\theta}$, $0 \leqslant \theta \leqslant \pi$, $\rho < t$, we obtain after some calculations.

$$\phi'(z) = -\sum_{n=0}^{\infty}\left[A_{2n} Z^{2n} + C_n Z^{-n-2}\right], \phi''(z) =$$

$$-\sum_{n=0}^{\infty}\left[(2n+2)\,A_{2n+2}\,Z^{2n+1} - (n+2)\,C_n\,Z^{-n-3}\right]$$

where

$$A_{2n} = \int_a^b \frac{g(t^2)}{t^{2n+1}}\,dt.$$

Also we denote B_{2n} by

$$B_{2n} = \int_a^b \frac{h(t^2)}{t\,2n+1}\,dt.$$

Now, the boundary conditions (2.6) lead to the expressions.

$$\sum_{n=0}^{\infty}\left\{2(n+1)\,C_{2n} + d_{2n}\right\}\rho^{-2n-2}\cos 2n\theta - \sum_{n=2}^{\infty} 2(n+1)$$

$$C_{2n-2}\,\rho^{-2n}\cos 2n\theta = 2A_0 +$$

$$+\sum_{n=1}^{\infty}(2n-2)\left\{A_{2n-2}\,\rho^{2n-3} - A_{2n}\,\rho^{2n}\right\}\cos 2n\theta +$$

$$KB_0\left\{\tfrac{1}{2} - \log(\rho/t)\right\} -$$

$$- \frac{K}{2} \sum_{n=1}^{\infty} \left\{ B_{2n-2} \quad \rho^{2n-2} - \frac{(n-1)}{n} B_{2n} \rho^{2n} \right\} \cos 2n\,\theta +$$

$$+ K \sum_{n=0}^{\infty} (n+1)\; a_{2n}\; \rho^{-2n} \cos 2n\,\theta ,$$

$$\sum_{n=1}^{\infty} \left\{ 2\,(n+1)\, C_{2n} + d_{2n} \right\} \rho^{-2n-2} \quad \sin 2n\theta - \sum_{n=2}^{\infty} 2n\; C_{2n-2}\; \rho^{-2n} \quad \sin 2n\,\theta =$$

$$= - \sum_{n=1}^{\infty} \left\{ (2n-2)\; A_{2n-2} \quad \rho^{2n-2} - 2n\; A_{2n}\; \rho^{2n} \right\} \sin\; 2n\theta -$$

$$- \frac{K}{2} \sum_{n=1}^{\infty} \left\{ B_{2n}\, \rho^{2n} - B_{2n-2} \quad \rho^{2n-2} \right\} \sin\; 2n\,\theta +$$

$$+ K \sum_{n=1}^{\infty} n \quad a_{2n} \quad \rho^{-2n} \;' \sin 2n\;\theta ,$$

for the determination of the coefficients C_{2n} and d_{2n}. In deriving these relations we have used the results (5—8) of the appendix. From these expressions, we observe thas

$$(\,2\;\; C_0 \; + \; d_0\,) \quad \rho^{-2} \;=\; 2A_0 + K \;\left[\; a_0 + B_0 \;\left\{ \tfrac{1}{2} - \log\,(\rho/t) \right\} \right] \qquad (5.6)$$

and for $n \geqslant 1$, we get

$$2\; C_{2n-2} \quad \rho^{-2n} \;= 2\; (2n-1) \quad A_{2n} \quad \rho^{2n} - 2\; (2n-2) \quad A_{2n-2} \quad \rho^{2n-2} \quad +$$

$$+ \frac{K}{2} \left\{ 2\; B_{2n-2} \quad . \;\rho^{2n-2} - \frac{1(n-1)}{n} B_{2n}\, \rho^{2n} \right\} - K\; a_{2n}\, \rho^{-2n} , \qquad (5.6)$$

$$\left\{ \;2\;\;(n+1)\;\; C_{2n} + \; d_{2n} \right\} \rho^{-2n-2} \;= 4n^2\;\; A_{2n} \quad \rho^{2n} - 2\;(n-1)\;\;(2n+1)$$

$$A_{2n-2}\ \rho^{2n-2} + \frac{K}{2}\left\{ (2n+1)\ B_{2n-2}\ \rho^{2n-2} - 2n\ B_{2n}\ \rho^{2n} \right\}. \qquad (5.7)$$

Substituting these values of the coefficients in the expression (5.2) for $G\ (r)$, we obtain that

$$G\ (r) = -\ \frac{2}{\pi}\left[\ 2\int_a^b \frac{g\ (x^2)}{x}\ K_1\ (x^2, r^2)\ dx + \frac{K}{2}\int_a^b \frac{h\ (x^2)}{x} K_2\ (x^2, r^2)\, dx \right] \qquad (5.8)$$

where

$$K_1\ (x^2, r^2) = \frac{\rho^2}{r^2} + \sum_{n=1}^{\infty} \frac{\rho^2}{r^2}\left(\frac{\rho^2}{rx} \right)^{2n}\left[4n^2 - n\ \rho^2\left(\frac{2n+3}{r^2} + \frac{2n+1}{x^2} \right) \right]$$

and

$$K_2\ (x^2, r^2) = -\log\ (\rho/x) + \frac{\rho^2}{r^2} + \sum_{n=1}^{\infty} \frac{x^2}{r^2}\left(\frac{\rho^2}{rx} \right)\left[\ (2n+1) - 4n\ (\rho^2/x^2) + \right.$$

$$\left. + \frac{2n^2+6n+1}{2\ (n+1)}\left(\rho^4/x^4 \right) \right].$$

Whenever $\rho << b$, the kernels $K_1\ (x^2, r^2)$ and $K_2\ (x^2,\ r^2)$ can be approximated to

$$K_1\ (x^2, r^2) = \frac{\rho^2}{r^2} + \frac{4\ \rho^6}{r^4 x^2} - \frac{\rho^8}{r^4 x^2}\left(\frac{5}{r^2} + \frac{3}{x^2} \right) + 0\ (\rho^{10}) \qquad (5.9)$$

$$K_2\ (x^2, r^2) = -\log\ (\rho/x) + \frac{\rho^2}{r^2} + \frac{3\ \rho^4}{r^4} - \frac{4\ \rho^6}{r^4 x^2} + 0\ (\rho^8)\,. \qquad (5.10)$$

We now consider the equation (5.4). We evaluate the arbitrary constant C from the condition $\int_a^b g\ (t^2)\ dt = 0.$ Its value is

$$C = \frac{1}{F} \int_a^b H\Big[\ G\ (r)\ \Big]\ dt$$

and

$$g\ (t^2) = -H\Big[\ G\ (r)\Big] + \frac{1}{FS\ (t^2)} \int_a^b H\Big[\ G\ (r)\ \Big]\ dt \tag{5.11}$$

where $F = F\ (\pi/2,\ q)$, the complete elliptic integral of the first kind and $S\ (t^2) = \{\ (t^2 - a^2)\ \ (b^2 - t^2)\ \}^{\frac{1}{2}}$. Substituting the values of $G\ (r)$ from (5.8) in (5·11) and using the results (9, 10, 11) of the appendix, we get a Fredholm integral equation of the second kind.

$$g\ (t^2) = K \int_a^b M_1\ (x^2,\ t^2) \frac{h\ (x^2)}{x}\ dx + \int_a^b M_2\ (x^2,\ t^2)\ \frac{g\ (x^2)}{x}\ dx, \tag{5.12}$$

where

$$M_1\ (x^2, t^2) = \frac{a\ b^3}{\pi\ S\ (t^2)} \Big[\ \frac{1}{a^3 b}\ (t^2 - \frac{b^2 E}{F})\ \log\ (\rho/x) + (\frac{1}{t^2} - \frac{E}{a^2 F})\rho^2 -$$

$$-\Big(\ 3\ \rho^4 - \frac{4\ \rho^6}{x^2}\Big)\ \Big\{\Big(\ \gamma + \frac{1}{a^2 t^2} - \frac{1}{t^4}\Big) +$$

$$+\ \delta\Big(\ \frac{1}{t^2} - \frac{E}{a^2 F}\ \Big)\ \Big\} + 0\ (\rho^8)\ \Big],$$

$$M_2\ (x^2, t^2) = \frac{4\ ab^3}{\pi S\ (t^2)} \Big[\ \Big(\ \frac{1}{t^2} - \frac{E}{a^2 F}\Big)\ \rho^2 - \frac{4\ \rho^6}{x^2} \Big\{\Big(\ \gamma + \frac{1}{a^2 t^2} - \frac{1}{t^4}\Big) +$$

$$+\ \delta\ \Big(\ \frac{1}{t^2} - \frac{E}{a^2 F}\ \Big)\Big\} + 0\ (\rho^8)\ \Big]$$

and

$$\gamma = \frac{1}{3a^4}\ \Big[\ \frac{a^2}{b^2}\Big(\ \frac{2E}{F} - 1\Big) - \frac{E}{F}\ \Big],\ \delta = \frac{3\ (a^2 - b^2)}{2a^2 b^2}.$$

The first term on the right hand side of (5.12) is the free term of the integral equation since $h(x^2)$ is a known function. The integral equation (5.12) can be easily seen to have weak singularities at the end points. Whatever has been said in section 4 regarding the solution of a similar Fredholm integral equation (4.10) will be valid for (5.12). To show $M_2(x^2, t^2)$ is bounded. it will be sufficient if we show that $K_1(x^2, t^2)$ is bounded. We express $K_1(x^2, t^2)$ as

$$K_1(x^2, t^2) = \frac{\rho^2}{t^2} + \frac{4\rho^2 z(1+z^2)}{t^2(1-z^2)^3} - \frac{\rho^4 z^2(5-z^2)}{t^4(1-z^2)^3} - \frac{z^4(3-z^2)}{(1-z^2)^2} \tag{5.13}$$

where $(\rho^2/t\,x) = z < 1$. The series on the right of (5.13) is easily seen to be convergent. Hence the solution of the Fredholm integral equation (5.12) can be found by the method of successive approximations. Let the first approximation be

$$g_0(t^2) = K\int_a^b \frac{h(x^2)}{x} M_2(x^2, t^2)\,dx$$

$$= \frac{PK}{2a^2 S(t^2)}\Bigg[\left(t^2 - \frac{b^2E}{F}\right)(p_1 + p_2\rho^4) + \left(\frac{1}{t^2} - \frac{E}{a^2F}\right)(p_3\rho^2 + p_4\rho^4 +$$

$$+ p_5\rho^6) + \beta_1\left(\gamma + \frac{1}{a^2t^2} - \frac{1}{t^4}\right)(3a^2b^2\rho^4 - 2(a^2+b^2)\rho^6) + 0(\rho^8)\Bigg], \tag{5.14}$$

where

$$p_1 = \frac{2(1-\alpha_1)b^2\beta_1}{a}, \quad p_2 = \frac{2(\beta_2 + \alpha_1\alpha_2\beta)\,p_1}{\beta_1} +$$

$$+ \frac{\alpha_1\alpha_2\beta\{4(1-\alpha_1)ab(a^2+b^2) + \alpha(a-b)^2\}}{2a^2\alpha}$$

$$p_3 = a^2b^2\beta_1, \; p_4 = 3a^2b\,\beta_1\,\delta, \; p_5 = a^2b^2(\beta_2 + \alpha_1\alpha_2\beta) + ab(a^2+b^2)\alpha_1\alpha_2 -$$

$$- \frac{2(a^2+b^2)\,\delta\,\beta_1}{b}.$$

The second approximation is obtained by putting (5.14) on the right fo (5.12). Hence

$$g_1(t^2) = g_0(t^2) + \int_a^b \frac{g_0(x^2)}{x} M_1(x^2, t^2)\, dx$$

$$= \frac{KP}{2a^2 S(t^2)} \Big[\left(t^2 - \frac{b^2E}{F} \right) (p_1 + p_2 \rho^4) + \left(\frac{1}{t^2} - \frac{E}{a^2F} \right) +$$

$$+ \left\{ (p_3 + 2b^2 q_1) \rho^2 + (p_4 + 2b^2 q_2) \rho^4 + (p_5 + 2b^2 q_3) \rho^6 \right\} +$$

$$+ \left(\gamma + \frac{1}{a^2t^2} - \frac{1}{t^4} \right) \left\{ 3p_3 \rho^4 + (2b^2 q_4 - 2\beta_1(a^2+b^2) \rho^6 \right\} + 0(\rho^8) \Big],$$

(5.15)

where

$$q_1 = \left(ab - \frac{b^2E}{F} \right) p_1, \quad q_2 = \left\{ \frac{a^2+b^2}{2a^2b^2} - \frac{E}{a^2F} \right\} p_3$$

$$q_1 = \frac{q_1 p_2}{p_1} + \frac{q^2 p_4}{p_3} + \frac{3\beta_1}{4a^2b^2} \{4a^4b^4 \gamma + 2b^2(a^2 + b^2) -$$

$$-3(a^4 + b^4) - a^8 b^8\} - \frac{2p_1 \delta}{a^2b} \{2a^2 - (a^2 + b^2) E/F\},$$

$$q_4 = \frac{2p_1}{a^2} \{(a^2 + b^2) E/F - 2a^2\}$$

The higher successive approximations can be continued similarly.

6- QUANTITIES OF PHYSICAL INTEREST :

Stress intensity factors :

The stress intensity factors at the two ends of a crack are given by

$$\left.\begin{aligned} N_a &= \underset{r\to a^-}{Lt}\ (a-r)^{\frac{1}{2}} \left[\ \sigma_{\theta\theta}\,(r,0)\ \right] \quad 0 < r < a \\ N_b &= \underset{r\to b^+}{Lt}\ (r-b)^{\frac{1}{2}} \left[\ \sigma_{\theta\theta}\,(r,0)\ \right] \quad r > b. \end{aligned}\right\} \qquad (6.1)$$

The equation (3.4) together with expressions for $\phi\ (z), \omega\ (z), T\ (r,\theta)$ and $V\ (r,\theta)$ and their corresponding derivatives, yields on separating the real part

$$\sigma_{\theta\theta}\,(r,0) = -2\int_a^b \frac{tg\ (t^2)}{t^2-r^2}\,dt - K\int_a^q \frac{\log\,|\,1-t^2/r^2\,|\ h\ (t^2)}{t}\,dt +$$

$$+ 2\int_a^b \frac{g\ (t^2)}{t}\,K_1\ (t^2, r^2)\ dt + \frac{K}{2}\int_a^b \frac{h\ (t^2)}{t} K_3\ (t^2,r^2)dt,$$

where $K_1\ (t^2, r^2)$ is given by (5.9) and

$$K_3\ (t^2, r^2) = -\log\ (\rho/t) + \rho^2/r^2 + \sum_{n=1}^{\infty} \left\{ \frac{\rho^2}{rt} \right\}^{2n} \left[\frac{n+1}{2n} + \right.$$

$$\left. + (2n+1)\ \frac{t^2}{r^2} - 4n\ (\rho^2/r^2) + \frac{2n^2+5n+1}{2\ (n+1)}\ \frac{\rho^4}{r^2\ t^2} \right].$$

For $0 < r \leqslant a$, we have

$$-2\int_a^b \frac{t\ g\ (t^2)}{t^2-r^2}\,dt = \frac{-KP\pi}{a^2}\left[\ (p_1+p_2\ \rho^4)\ \left\{1 + \frac{r^2 - b^2\ E/F}{R}\right\} + \right.$$

$$+ \ \{(p_3+2b^2q_1)\ \rho^2 + (p_4+2b^2\ q_2)\ \rho^4 + (p_5 + 2b^2\ q_3)\ \rho^6\} +$$

$$+ \ \left\{\frac{1}{R}\left(\frac{1}{r^2} - \frac{E}{a^2F}\right) - \frac{1}{abr^2}\right\} +$$

$$+ \{3p_3 \rho^4 + (2b^2 q_4 - 2 \beta_1 (a^2 + b^2) \rho^6\} \left\{ \frac{1}{R} (r + a^{-2} r^{-2} - r^{-4}) + \right.$$

$$\left. + \frac{1}{r^2 ab} \left(\frac{1}{r^2} + \frac{(a^2+b^2)}{2a^2 b^2} - \frac{1}{a^2} \right) \right\} + O(\rho^8) \Bigg], \qquad (6.2)$$

and

$$-K \int_a^b \frac{\log | 1 - t^2/r^2 | h(t^2)}{t} dt = \frac{-PK\pi}{2ab} \Bigg[2 \{\beta_1 + (\beta_2 + \alpha_1 \alpha_2 \beta) \rho^4\} \times$$

$$\times \log \frac{a\sqrt{(b^2-r^2)} + b\sqrt{(a^2-r^2)}}{r(a+b)} + \left(\frac{R}{r^2} - \frac{ab}{r^2} - \frac{b}{a} + 2 + \right.$$

$$\left. + \frac{(a^2+b^2)}{2ab} \left\{ 2 \log \left(\frac{b-a}{b+a} \right) + \log \frac{a^2 (b^2-r^2) + b^2 (a^2-r^2) + 2ab R}{r^2 (b^2-a^2)} \right\} \right)$$

$$\alpha_1 \alpha_2 \rho^4 + O(\rho^8), \Bigg], \qquad (6.3)$$

where $R = \left\{ (a^2-r^2)(b^2-r^2) \right\}^{1/2}$. In deriving (6.2) and (6.3), we have used results (14, 15) of the appendix. Again, for $r \geqslant b$, we have

$$-2 \int_a^b \frac{t g(t^2)}{t^2-r^2} dt = -\frac{KP\pi}{2a^2} \Bigg[(p_1 + p_2 \rho^4) \left\{ 1 - \frac{r^2 - b^2 E/F}{R_1} \right\} +$$

$$+ \left\{ (p_3 + 2b^2 q_1) \rho^2 + (p_4 + 2b^2 q_2) \rho^4 + (p_5 + 2b^2 q_3) \rho^6 \right\}$$

$$\left\{ \frac{1}{R_1} \left(\frac{E}{a^2 F} - \frac{1}{r^2} \right) - \frac{1}{abr^2} \right\} +$$

$$+ \{3p_3\,\rho^4 + (2b^2 q_4 - 2\,\beta_1\,(a^2+b^2)\,\rho^6\} \left\{ \frac{1}{R_1} \quad (r - a^{-2}\, r^{-2} + r^{-4} \quad + \right.$$

$$\left. + \frac{1}{abr^2}\left(r^{-2} + \frac{a^2+b^2}{2a^2\,b^2} - a^{-2}\right) \right\} + O\,(\rho^8) \Big] , \qquad (6{,}4)$$

and

$$-K \int_a^b \frac{\log|1-t^2/r^2|\,h\,(t^2)}{t}\,dt = -\frac{KP\pi}{2ab}\Big[\; 2\,\{\,\beta_1 + (\beta_2 + \alpha_1\alpha_2\beta)\,\rho^6\}\,\times$$

$$\log \frac{a\,\sqrt{(r^2-b^2)} + b\,\sqrt{(r^2-a^2)}}{r\,(a+b)} + \Bigg(2 - \frac{a}{b} - \frac{ab}{r^2} - \frac{R_1}{r^2} + \frac{(a^2+b^2)}{2ab}$$

$$\left\{ 2 \log \left\{ \frac{b-a}{b+a} \right\} + \log \frac{a^2\,(r^2-b^2) + b^2\,(r^2-a^2) + 2ab\,R_1}{r^2\,(b^2-a^2)} \right\} \Bigg)$$

$$\alpha_1\,\alpha_2\,\rho^4 + O\,(\,\rho^8\,)\Big] , \qquad (6.5)$$

where $R_1 = \sqrt{\{\,(r^2-b^2)\,(r^2-a^2)\,\}}$.

It is easy to show with the help of results of the appendix that

$$2\int_a^b \frac{g\,(t^2)}{t}\,K_1\,(r^2, t^2)\,dt = \frac{KP\pi}{2a^2\,r^2}\Big[\; p_1\,c_0\,\rho^2 + (p_3 + 2b^2\,q_1)\;C^2\,\rho^4 +$$

$$+ \left\{ (p_2\,c_0 + 4\,p_1\,c_1\,r^{-2}) + c_2\,(p_4 + 2b^2\,q_2) + 3a^2b^2\beta_1\,c_3 \right\}\rho^6 + O\,(\rho^8) \Big]$$

(6.6)

where

$$c_0 = 1 - \frac{bE}{aF},\; c_1 = \frac{1}{ab}\Big[\; 1 - (1 + b^2/a^2)\; E/F \;\Big]$$

$$c_2 = \frac{1}{2a^3b}\left[\left(\frac{a^2}{b^2}+1\right) - \frac{2E}{F}\right], c_3 = \frac{1}{8a^5b^5}\left[+ \right.$$

$$\left. + 8\gamma a^4 b^4 + 2a^2b^2 + b^4 - 3a^4 \right];$$

and

$$\frac{K}{2}\int_a^b K_3(t^2, r^2)\ \frac{h(t^2)}{t}\ dt = \frac{KP\pi}{2}\left[d_0 + d_1\ \frac{\rho^2}{r^2} + (d_2 + a_2\beta_1 r^{-2} + \right.$$

$$\left. + 3d_1 r^{-4})\ \rho^4 + (d_3 r^{-2} - 4\ a_2\ \beta_1 r^{-4})\ \rho^6 + 0\ (\rho^8)\right], \tag{6.7}$$

where

$$d_0 = \frac{(a_1 - 1)\ \beta_1}{a}, d_1 = \frac{\beta_1}{2ab}, d_2 = (\beta_2 + a_1\ a_2\ \beta)\frac{d_0}{\beta_1} +$$

$$+ \frac{a_1 a_2}{ab}\left\{(a^2 + b^2))\ \frac{d_0}{\beta_1} + \frac{(a-b)^2}{4ab}\right\},$$

$$d_3 = \frac{1}{2a^2b^2}\ [a_1\ a_2\ (a^2+b^2) + ab\ (\beta_2 + a_1\ a_2\ \beta)\ ].$$

Hence the stress intensity factors as estimated from (6.1) are

$$\mathcal{N}_a = \frac{KP\pi}{2a^2\sqrt{2a\,(b^2-a^2)}}\left[(p_1 + p_2\ \rho^4)\left(\frac{b^2E}{F} - a^2\right) + \right.$$

$$+ \{ (p_3 + 2b^2\ q_1)\ \rho^2 + (p_4 + 2b^2\ q_2)\ \rho^4 + (p_5 + 2b^2\ q_3)\ \rho^6 \} +$$

$$+ \left(\frac{E}{F} - 1\right) a^{-2} - \gamma\left\{ 3\ \beta_1\ a^2b^2\ \rho^4 + \right.$$

$$\left.\left. + (2b^2\ q_4 - 2\ (a^2+b^2)\beta_1\ \rho^6 \right\} + 0\ (\rho^8)\right] \tag{6.8}$$

$$N_b = \frac{KP\pi}{2a^2\sqrt{2b\,(b^2-a^2)}}\Big[\; b^2\,(p_1 + p_2\,\rho^4)\Big(1-\frac{E}{F}\Big) +$$

$$+ \{ (p_3+2b^2\,q_1)\,\rho^2+(p_4+2b^2\,q_2)\,\rho^4 + (p_5+2b^2\,q_3)\,\rho^6\} \Big(\frac{1}{b^2}-\frac{E}{a^2F}\Big) +$$

$$+(\gamma - a^{-2}b^{-2} + b^{-4})\Big\{ 3\,\beta_1\,a^2\,b^2\,\rho^4 +$$

$$+ (2b^2\,q_4 - 2\,(a^4+b^4)\,\beta_1)\,\rho^6 \Big\} + 0\,(\rho^8)\Big]\,. \tag{6.9}$$

The normal displacement along the crack is given by

$$\mu\, u_\theta\,(r,\,0) = \frac{1+k}{2}\int_0^\infty B\,(\xi)\cos\,(\xi r)\;d\xi\,, \qquad a \leqslant r \leqslant b. \tag{6.10}$$

Substituting the value of $B\,(\xi)$ from (5.3) and performing the change of order of integration, we obtain

$$\mu\, u_\theta\,(r,\,0) = \frac{\pi\,(1+k)}{4}\int_r^b g\,(t^2)\;dt. \tag{6.11}$$

Hence

$$\mu u_\theta(r,\,0) = \frac{\pi PK\,(1+k)}{8a^4b}\Big[\Big\{E(\phi,\,q)-\frac{E}{F}\,F\,(\phi,q)\Big\}\Big\{a^2b^2p_1+$$

$$(p_3+2b^2q_1)\,\rho^2+(a^2b^2p_2+p_4+2b^2q_2)\,\rho^4+(p_5+2b^2q_3)\rho^6\Big\}+$$

$$+\frac{1}{3a^2b^2}\Big\{ (b^2-2a^2)\,E\,(\phi,\,q) + a^2(3a^2b^2\gamma+1)\,F\,(\phi,\,q)+$$

$$\frac{\{(r^2-a^2)\,(b^2-r^2)\}^{\frac{1}{2}}}{r^2}\Big\}\Big\{ 3\beta_1a^2b^2\rho^4+2(b^2q_4-(a^2+b^2)\beta_1)\rho^6\Big\} + 0\,(\rho^8)\Big] \tag{6.12}$$

where

$$\sin\phi = \Big\{(b^2-r^2)\,/\,(b^2-a^2)\Big\}^{\frac{1}{2}}.$$

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APPENDIX

It is simple to derive the following results.

$$\int_0^\infty e^{-\xi\rho\sin\theta}\left(\frac{1-\cos\xi t}{\xi}\right)\left.\begin{matrix}\cos(\xi\rho\cos\theta)\\ \sin(\xi\rho\cos\theta)\end{matrix}\right\}d\xi=\begin{cases}-\log(\rho/t)-\sum_{n=1}^{\infty}\left(\frac{\rho}{t}\right)^{2n}\frac{\cos 2n\theta}{2n}\\ (\pi/2-\theta+\sum_{n=1}^{\infty}\left(\frac{\rho}{t}\right)^{2n}\frac{\sin 2n\theta}{2n}\end{cases}\tag{1}$$

$$\int_0^\infty e^{-\xi\rho\sin\theta}(1-\cos\xi t)\left.\begin{matrix}\cos(\xi\rho\cos\theta)\\ \sin(\xi\rho\cos\theta)\end{matrix}\right\}d\xi=\sum_{n=0}^{\infty}\frac{\rho^{2n-1}}{t^{2n}}\left\{\begin{matrix}\sin(2n-1)\theta\\ \cos(2n-1)\theta\end{matrix}\right.\tag{2}$$

$$\int_0^\infty e^{-\xi\rho\sin\theta}\sin\xi t\left.\begin{matrix}\cos(\xi\rho\cos\theta)\\ \sin(\xi\rho\cos\theta)\end{matrix}\right\}d\xi=\sum_{n=0}^{\infty}\frac{\rho^{2n}}{t^{2n+1}}\left\{\begin{matrix}\cos 2n\theta\\ \sin 2n\theta\end{matrix}\right.\tag{3}$$

$$\int_0^\infty e^{-\xi\rho\sin\theta}\xi\sin\xi t\left.\begin{matrix}\cos(\xi\rho\cos\theta)\\ \sin(\xi\rho\cos\theta)\end{matrix}\right\}d\xi=\pm\sum_{n=0}^{\infty}\frac{2(n+1).\rho^{2n+1}}{t^{2n+3}}\left\{\begin{matrix}\sin(2n+1)\theta\\ \cos(2n+1)\theta\end{matrix}\right.\tag{4}$$

From these results we can easily show that

$$\int_0^\infty e^{-\xi\rho\sin\theta}(1-\cos\xi t)\,\rho\sin\theta\sin(2\theta+\xi\rho\cos\theta)\,d\xi =$$

$$\tfrac{1}{2}\sum_{n=1}^{\infty}\left\{\frac{\rho^{2n}}{t^{2n+1}}-\frac{\rho^{2n-1}}{t^{2n-1}}\right\}\sin 2n\theta \qquad (5)$$

$$\int_0^\infty e^{-\xi\rho\sin\theta}\left(\frac{1-\cos\xi t}{\xi}\right)\Big[\xi\rho\sin\theta\cos(2\theta+\xi\rho\cos\theta)-\cos(\xi\rho\cos\theta)\Big]d\xi =$$

$$t^{-1}\{\log(\rho/t)-\tfrac{1}{2}\}-\sum_{n=1}^{\infty}\left[\frac{n-1}{2n}\frac{\rho^{2n}}{t^{2n+1}}-\tfrac{1}{2}\frac{\rho^{2n-2}}{t^{2n-1}}\right]\cos 2n\theta, \qquad (6)$$

$$-\int_0^\infty \xi\rho e^{-\xi\rho\sin\theta}\sin\theta\sin(2\theta+\xi\rho\cos\theta)\sin\xi t\,d\xi =$$

$$=\sum_{n=1}^{\infty}\left[(2n-2)\frac{\rho^{2n-2}}{t^{2n-1}}-2n\frac{\rho^{2n}}{t^{2n+1}}\right]\sin 2n\theta \qquad (7)$$

$$-\int_0^\infty e^{-\xi\rho\sin\theta}\Big[(1-\xi\rho\sin\theta\cos 2\theta).\cos(\xi\rho\cos\theta)+\xi\rho\sin 2\theta\sin\theta$$

$$\sin(\xi\rho\cos\theta)\Big]\times\sin\xi t\,d\xi = -\frac{1}{t}-\sum_{n=1}^{\infty}(2n-2)\left[\frac{\rho^{2n-2}}{t^{2n-1}}-\frac{\rho^{2n}}{t^{2n+1}}\right].\cos 2n\theta \qquad (8)$$

Following results are well known and can be found in Gardshteyn and Ryzhik [4].

$$\int_a^b\{(t^2-a^2)(b^2-t^2)\}^{-\frac{1}{2}}\,dt = \frac{1}{b}F(\pi/2,q) = \frac{F}{b} \qquad (9)$$

$$\int_a^b t^2\{(t^2-a^2)(b^2-t^2)\}^{-\frac{1}{2}}\,dt = bE(\pi/2,q) = bE \qquad (10)$$

$$\int_a^b \frac{t\,dt}{\sqrt{\{(t^2-a^2)(b^2-t^2)\}}\,(t^2-r^2)} = \begin{cases} \pi/2\,\{(a^2-r^2)(b^2-r^2)\}^{-\frac{1}{2}}, & 0<r\leq a \\ 0, & a<r<b \\ \pi/2\,\{(r^2-a^2)(r^2-b^2)\}^{-\frac{1}{2}}, & r\geq b \end{cases} \tag{11}$$

Let $I_n = \int_0^{\pi/2} \frac{\log(a^2\sin^2\theta + b^2\cos^2\theta)}{(a^2\sin^2\theta+b^2\cos^2\theta)^n}\,d\theta$

Then

$$I_{n+1} = \frac{(a+b)}{nb}\int_0^{\pi/2}\frac{\sin^2\theta\,d\theta}{(a^2\sin^2\theta+b^2\cos^2\theta)^{n+1}} +$$

$$+\frac{(a+b)}{na}\int_0^{\pi/2}\frac{\cos^2\theta\,d\theta}{(a^2\sin^2\theta+b^2\cos^2\theta)^{n+1}} -$$

$$-\frac{I_n}{ab}-\frac{(a+b)}{2nab}\left(\frac{\partial I_n}{\partial a}+\frac{\partial I_n}{\partial b}\right), \tag{12}$$

and

$$\int_0^{\pi/2}\frac{\log(a^2\sin^2\theta+b^2\cos^2\theta)}{a^2\sin^2\theta+b^2\cos^2\theta}\,d\theta = \frac{\pi}{ab}\log\frac{2ab}{a+b}. \tag{13}$$

Following results can be easily derived from (12) and (13)

$$\int_a^b \frac{\log|1-t^2/r^2|}{t\sqrt{\{(t^2-a^2)(b^2-t^2)\}}} = \begin{cases} \frac{\pi}{ab}\left[\log\frac{a\sqrt{(b^2-r^2)}+b\sqrt{(a^2-r^2)}}{r(a+b)}\right], & 0<r\leq a \\ \frac{\pi}{2ab}\log\frac{b-a}{b+a}, & a<r<b \\ \frac{\pi}{ab}\log\frac{a\sqrt{(r^2-b^2)}+b\sqrt{(r^2-a^2)}}{r(a+b)}, & r\geq b \end{cases} \tag{14}$$

$$\int_a^b \frac{\log |1-t^2/r^2|}{t^3\sqrt{\{(t^2-a^2)(b^2-t^2)\}}}dt = \begin{cases} \frac{\pi}{2a^2b^2}\left[1+\frac{R-ab}{r^2}\right]+\frac{\pi(a^2+b^2)}{4a^3b^3}+ \\ +\left[\log\frac{b-a}{b+v}+\right. \\ \left.+\log\frac{a^2(b^2-r^2)+b^2(a^2-r^2)+2abR}{r^2(b^2-a^2)}\right], \\ \qquad 0<r\leqslant a \\ \frac{\pi}{2a^2b^2}\left[1-\frac{ab}{r^2}\right]+\frac{\pi(a^2+b^2)}{4a^3b^3} \\ \log\frac{b-a}{b+a}, \qquad a<r<b \\ \frac{\pi}{2a^2b^2}\left[1-\frac{R_1+ab}{r^2}\right]+\frac{\pi(a^2+b^2)}{4a^3b^3} \\ \left[\log\frac{b-a}{b+a}+\right. \\ +\log\frac{a^2(r^2-b^2)+b^2(r^2-a^2)+2ab\,R_1}{r^2(b^2-a^2)}, \\ \qquad r\geqslant b \end{cases} \quad (15)$$

where

$R=\surd\{(a^2-r^2)(b^2-r^2)\}$ and $R_1=\surd\{(r^2-a^2)(r^2-b^2)\}$.