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ON A CLASS OF GENERALIZED HYPERGEOMETRIC DISTRIBUTIONS*

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SUMMARY

In the course of an attempt to present a unified theory of the classical probability distributions, the authors introduce and study here a general family of statistical probability distributions involving Fox's H-function. In particular, the distribution function, the characteristic function, distributions of the largest order statistic and of the ratio of two independent stochastic variables having the probability functions in this family of statistical probability distributions, etc., are investigated.

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Several interesting properties of this class of generalized hypergeometric distributions are also pointed out.

1. INTRODUCTION.

In the literature on probability theory a large number of statistical distributions have been studied, in varying details, from time to time because of their enormous practical applications. Indeed there are frequent instances of studies of some general classes of statistical distributions, such as the general hypergeometric distribution, the generalized beta and gamma distributions, and so on. Recently, MATHAI and SAXENA [4] have introduced what they call a generalized hypergeometric distribution whose probability density function involves the Gaussian ordinary hypergeometric function ${}_2F_1$ (see, e. g., [1], p. 56). A limiting form of this probability distribution would lead to what is well-known in the literature as the general hypergeometric distribution the density function of which involves Kummer's confluent hypergeometric function ${}_1F_1$ [loc. cit., p. 248]. Thus it is readily seen that almost all classical statistical distributions, such as the generalized beta and gamma distributions, the generalized F-distribution, Student's t-distribution, the normal distribution, the exponential distribution, the waiting time distribution, the logistic distribution, and the distributions that go with the names of Cauchy, Raleigh and Weibull, can be derived as specialized or limiting cases of the so-called generalized hypergeometric distribution.

The motivation of the present paper is manifold. A critical analysis of the aforementioned work of Mathai and Saxena would reveal, among other things, the fact that it is only the generalized F-distribution which follows as a particular case of the distribution they have studied. In order to deduce the other classical statistical distributions from theirs, they obviously have had to recourse to a certain limiting process by means of which their generalized hypergeometric distribution would reduce, first of all, to the known hypergeometric distribution involving Kummer's ${}_1F_1$ function. Also in the study of the characteristic function and of the distribution of the ratio of two independent stochastic variables with probability functions expressed in terms of Gauss' ${}_2F_1$, they have to bring in a wider class of functions, viz. Fox's H-function defined by [2, p. 408]

$$(1) \quad H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_1, A_1), \ldots, (a_p, A_p) \\ \\ (b_1, B_1), \ldots, (b_q, B_q) \end{matrix} \right]$$

$$= \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^{m} \Gamma(b_j - B_j \zeta) \prod_{j=1}^{n} \Gamma(1 - a_j + A_j \zeta)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + B_j \zeta) \prod_{j=n+1}^{p} \Gamma(a_j - A_j \zeta)} z^{\zeta} \, d\zeta,$$

where an empty product is interpreted as unity, $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$, the A_j and B_j are all positive, the poles of the integrand of (1) are simple, C is a suitable contour of Mellin-Barnes' type which runs from $\tau - i\infty$ to $\tau + i\infty$ with indentations, if necessary, in such a manner that all the poles of $\Gamma(b_j - B_j\zeta)$, $j = 1, \ldots, m$, are to the right, and those of $\Gamma(1 - a_j + A_j\zeta)$, $j = 1, \ldots, n$, to the left, of C, and the integral in (1) converges if

$$(2) \qquad |\arg(z)| < \tfrac{1}{2} \pi \Delta,$$

with

$$(3) \qquad \Delta \equiv \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j > 0.$$

These conditions will be assumed throughout the present paper, and for convenience, we shall abbreviate the first member of (1) by

$$H_{p,q}^{m,n} [z].$$

Evidently

$$(4) \quad H_{p,q}^{m,n} [z] = H_{q,p}^{n,m} \left[\frac{1}{z} \,\middle|\, \begin{matrix} (1-b_1, B_1), \ldots, (1-b_q, B_q) \\ (1-a_1, A_1), \ldots, (1-a_p, A_p) \end{matrix} \right],$$

$$(5) \quad z^{\delta} H_{p,q}^{m,n} [z] = H_{p,q}^{m,n} \left[z \,\middle|\, \begin{matrix} (a_1 + \delta A_1, A_1), \ldots, (a_p + \delta A_p, A_p) \\ (b_1 + \delta B_1, B_1), \ldots, (b_q + \delta B_q, B_q) \end{matrix} \right]$$

and

$$(6) \quad H_{p,q}^{m,n} \left[z \,\middle|\, \begin{matrix} (a_1, 1), \ldots, (a_p, 1) \\ (b_1, 1), \ldots, (b_q, 1) \end{matrix} \right] = G_{p,q}^{m,n} \left(z \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \right),$$

where $G_{p,q}^{m,n}(z)$ denotes the G-function of MEIJER [5].

It may be of interest to note that since [1, p. 215]

$$(7) \quad {}_pF_q \left[\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} z \right] = \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} \cdot G_{p,q+1}^{1,p} \left(-z \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_p \\ 0, 1-b_1, \ldots, 1-b_q \end{matrix} \right),$$

both Gauss' ${}_2F_1$ and Kummer's ${}_1F_1$ can be recovered from the G-function, and hence also from the H-function, by merely specializing the parameters appropriately, there being no limiting processes involved. Moreover, Meijer's G-functions given by (6) do find several interesting applications in statistical distribution problems (cf., e. g., [3]). Thus it would seem quite natural to consider, in this paper, a general family of statistical probability distributions having the probability density function

$$(8) \quad p(x) = U\left[\alpha, \beta, \gamma : (a_p, A_p)_n ; (b_q, B_q)_m \right] x^{\beta-1} H_{p,q}^{m,n} \left[\alpha x^{\gamma} \right],$$

for $x > 0$, $\gamma > 0$,

$$(9) \quad -\min_{1 \leqslant j \leqslant m} \left(\frac{b_j}{B_j} \right) < \frac{\beta}{\gamma} < -\max_{1 \leqslant j \leqslant n} \left(\frac{a_j - 1}{A_j} \right),$$

and $p(x) = 0$ elsewhere ; wherein, for convenience,

$$(10) \quad U\left[\alpha, \beta, \gamma : (a_p, A_p)_n ; (b_q, B_q)_m \right]$$

$$= \frac{\gamma \, \alpha^{\beta/\gamma} \prod_{j=m+1}^{q} \Gamma\left(1 - b_j - \frac{\beta}{\gamma} B_j\right) \prod_{j=n+1}^{p} \Gamma\left(a_j + \frac{\beta}{\gamma} A_j\right)}{\prod_{j=1}^{m} \Gamma\left(b_j + \frac{\beta}{\gamma} B_j\right) \prod_{j=1}^{n} \Gamma\left(1 - a_j - \frac{\beta}{\gamma} A_j\right)},$$

it being understood that the parameters involved are so restricted that $p(x)$ remains positive.

For several interesting properties of the H-function, in addition to relationships (4) and (5) above, one may refer, for instance, to FOX [2, pp. 408-429] and SRIVASTAVA and DAOUST [7, pp. 451-453]. Indeed its special case, the G-function, has been treated extensively, for instance, by MEIJER [5] and ERDÉLYI et al. [1, pp. 206-222].

It seems worthwhile to remark in passing that in our systematic study of the generalized statistical probability distribution associated with (8) we do not encounter any higher transcendental functions other than the H-functions themselves.

2. THE CHARACTERISTIC FUNCTION.

The characteristic function of $p(x)$ is given by

$$\phi(t) = E[e^{itX}] = \int_0^\infty e^{itx}\, p(x)\, dx, \tag{11}$$

where $i = \sqrt{-1}$, and E stands for 'mathematical expectation'.

On substituting for $p(x)$ from (8), if we replace the H-function by its contour integral (1), invert the order of integration, and then evaluate the inner Eulerian integral, we shall obtain the characteristic function as

$$\phi(t) = U[\alpha, \beta, \gamma : (a_p, A_p)_n ; (b_q, B_q)_m] \cdot (-it)^{-\beta} H^{m, n+1}_{p+1, q}\left[\alpha(-it)^{-\gamma} \,\middle|\, \begin{matrix} (1-\beta, \gamma), (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{matrix}\right], \tag{12}$$

where $U[\ldots]$ is given by (10).

The derivatives of the H-function, occurring on the right-hand side of (12), can be expressed fairly easily in terms of H-functions themselves. Thus it would be quite straightforward to evaluate the moments and related measures for the general family of probability distributions defined by (8).

3. THE DISTRIBUTION FUNCTION.

The distribution function $F(x)$ or the cumulative probability function for the probability density function $p(x)$ is given by

$$F(x) = \int_{-\infty}^{x} p(t)\, dt \tag{13}$$

$$= U[\alpha, \beta, \gamma : (a_p, A_p)_n ; (b_q, B_q)_m] \int_0^x t^{\beta-1} H_{p,q}^{m,n} \left[a t^{\gamma} \right] dt.$$

Now substitute the contour integral for the H-function and change the order of integration. Then evaluate the inner integral and interpret the resulting expression in terms of the H-function.

We thus find that

$$F(x) = U\left[\alpha, \beta, \gamma : (a_p, A_p)_n ; (b_q, B_q)_m \right]$$

(14)

$$. x^{\beta} H_{p+1, q+1}^{m, n+1} \left[a x^{\gamma} \left| \begin{matrix} (1-\beta,\gamma), (a_1, A_1), ..(a_p, A_p) \\ (b_1, B_1),..., (b_q, B_q), (-\beta,\gamma) \end{matrix} \right. \right]$$

4. DISTRIBUTION OF THE LARGEST ORDER STATISTIC.

The density function for the largest order statistic for a sample of size N from a population $f(x)$ is given by

$$(15) \qquad g(x) = N \left[\int_{-\infty}^{x} f(t)\, dt \right]^{N-1} . f(x).$$

Thus if we let $f(x)$ be the same as $p(x)$ defined by (8), then the density function will be given by

$$(16) \qquad g(x) = N\, U[\alpha, \beta, \gamma : (a_p, A_p)_n ; (b_q, B_q)_m]$$

$$. x^{\beta-1} \{F(x)\}^{N-1} H_{p,q}^{m,n} \left[a x^{\gamma} \right],$$

where $F(x)$ is defined by (14) above.

The special case of (16) when $N = 2$ is worthy of note. Indeed, for a sample of size 2, the density function $g(x)$ is expressible as the product of two H-functions having the same argument.

5. DISTRIBUTION OF THE RATIO.

Consider the distribution of the ratio of two independent stochastic variables X and Y whose probability density functions $p(x)$ and $q(y)$ belong to the same family as in (8) above. Thus if we suppose $p(x)$ to be given by (8) itself, we may write analogously

(17) $$q(y) = U\left[\lambda, \rho, \sigma : (e_r, E_r)_\nu ; (g_s, G_s)_\mu\right]$$

$$\cdot y^{\sigma-1} H^{\mu,\nu}_{r,s}\left[\lambda y^\sigma \middle| \begin{matrix} (e_1, E_1), \ldots, (e_r, E_r) \\ (g_1, G_1), \ldots, (g_s, G_s) \end{matrix}\right],$$

for $y > 0$, $\sigma > 0$,

(18) $$-\min_{1 \leqslant j \leqslant \mu}\left(\frac{g_j}{G_j}\right) < \frac{\rho}{\sigma} < -\max_{1 \leqslant j \leqslant \nu}\left(\frac{e_j - 1}{E_j}\right),$$

and $q(y) = 0$ elsewhere ; where

(19) $$\Omega \equiv \sum_{j=1}^{\nu} E_j - \sum_{j=\nu+1}^{r} E_j + \sum_{j=1}^{\mu} G_j - \sum_{j=\mu+1}^{s} G_j > 0,$$

it being understood, as before, that the parameters involved are so constrained that $q(y)$ remains positive.

If we put $W = X/Y$, then

(20) $$V = \log W = \log X - \log Y,$$

and the characteristic function for V is given by

$$\phi_V(t) = E\left[e^{itV}\right] = \int_0^\infty \int_0^\infty e^{it(\log x - \log y)} p(x) q(y)\, dx\, dy$$

(21) $$= \int_0^\infty x^{it} p(x)\, dx . \int_0^\infty y^{-it} q(y)\, dy.$$

On substituting for $p(x)$ and $q(y)$ from (8) and (17) respectively, if we evaluate the resulting integrals in (21) by making an appeal to the Mellin transform of the H-function [cf., e. g., [7], p. 452, formula (2.5)], we shall at once arrive at the elegant result

(22) $$\phi V(t) = \frac{U\left[\alpha, \beta, \gamma : (a_p, A_p)_n ; (b_q, B_q)_m\right]}{U\left[\alpha, \beta + it, \gamma : (a_p, A_p)_n ; (b_q, B_q)_m\right]}$$

$$\cdot \frac{U\left[\lambda, \rho, \sigma : (e_r, E_r)_\nu ; (g_s, G_s)_\mu\right]}{U\left[\lambda, \rho\text{-}it, \sigma : (e_r, E_r)_\nu ; (g_s, G_s)_\mu\right]}.$$

Now the Fourier transform of $\phi_V(t)$ would give us the density function $p_V(x)$ of V. Thus

$$p_{\hat{V}}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi_V(t)\,dt. \tag{23}$$

If we substitute for $\phi_V(t)$ from (22) and put $\rho - i\,t = \zeta$, then the second member of (23) will transform itself into a contour integral of Mellin-Barnes' type which can readily be interpreted as an H-function by means of (1).

In order to obtain the density function $p_W(x)$ of W we apply the transformation $W = e^V$, and we finally have

$$p_W(x) = U[\alpha, \beta, \gamma : (a_p, A_p)_n ; (b_q, B_q)_m] \tag{24}$$

$$\cdot\, U\left[\lambda, \rho, \sigma : (e_r, E_r)_\nu ; (g_s, G_s)_\mu\right] \frac{x^{-\rho}\,\alpha^{-(\beta+\rho)/\sigma}}{\gamma\sigma}$$

$$\cdot H^{m+\nu,\, n+\mu}_{p+s,\, q+r}\left[\frac{x\,\alpha^{1/\gamma}}{\lambda^{1/\sigma}}\,\middle|\,\begin{matrix}\left(\delta_1, \frac{A_1}{\gamma}\right),\ldots,\left(\delta_n, \frac{A_n}{\gamma}\right)\left(1\text{-}g_1, \frac{G_1}{\sigma}\right),\ldots,\left(1-g_s, \frac{G_s}{\sigma}\right),\left(\delta_{n+1}, \frac{A_{n+1}}{\gamma}\right),\ldots,\left(\delta_p, \frac{A_p}{\gamma}\right)\\ \left(\omega_1, \frac{B_1}{\gamma}\right),\ldots,\left(\omega_m, \frac{B_m}{\gamma}\right)\left(1 - e_1, \frac{E_1}{\sigma}\right),\ldots,\left(1-e_r, \frac{E_r}{\sigma}\right),\left(\omega_{m+1}, \frac{B_{m+1}}{\gamma}\right),\ldots,\left(\omega_q, \frac{B_q}{\gamma}\right)\end{matrix}\right],$$

for $x > 0$ and $p_W(x) = 0$ elsewhere; where, for the sake of brevity,

$$\begin{cases} \delta_j = a_j + \left(\frac{\beta+\rho}{\gamma}\right) A_j,\ j = 1,\ldots, p, \\ \omega_j = b_j + \left(\frac{\beta+\rho}{\gamma}\right) B_j,\ j = 1,\ldots, q. \end{cases} \tag{25}$$

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