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ON A FIXED POINT THEOREM

By

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ABSTRACT

The aim of the present paper is to obtain a generalization of a recent result due to Pant et al. Our theorem extends the scope of the study of common fixed point theorems from the class of compatible maps to the wider class of pointwise R-weakly commuting maps.

1. Introduction. Two selfmaps A and S of a metric space (X, d) are called compatible if $lim_n\ d\ (ASx_n, SAx_n) = 0$ whenever $\{X_n\}$ is a sequence such that $lim_n\ AX_n = lim_n\ Sx = t$ for some t in X. In [6], the first author obtained another generalization of weak commutativity by introducing the concept of R-weak commutativity. Two self maps A and S of a metric space (X,d) are called pointwise R-weakly commuting on X if given x in X there exists $R > 0$ such that $d(ASx, SAx) \leq Rd(Ax, Sx)$. It is obvious that A and S can fail to be pointwise R-weakly commuting only if there exists some x in X such that $Ax = Sx$ but $ASx \neq SAx$, that is, only if they possess a coincidence point at which they do not commute. Thus pointwise R-weakly commuting maps may equivalently be defined as coincidence preserving maps and we shall use these terms interchangeably. It follows easily from this, as is formally eastablished in [11], that pointwise R-weak commutativity is a necessary, hence minimal condition for the existence of common fixed points of contractive type maps. Moreover, compatible maps are necessarily coincidence preserving since compatible maps commute at coincidence points.

Recently, employing the notion of pointwise *R-weak* commutativity Pant et al [12], proved the following :

Theorem (Pant et al. [12]). *Let* $\{A_i\}\ i = 1,2,3, \ldots$, S and T be selfmappings of a complete metric space (X,d) such that

(i) given u in X there exists y, v in X such that $A_1u = Ty$ and $A_2y = Sv$,

(ii) $d(A_1x, A_iy) < M_{1i}(x,y)$ for all x,y in X whenever $M_{1i}(x,y) > 0$ and $A_1x = A_iy$ whennever $A_1x = Sx$ and $A_iy = Ty$,

(iii) $d(A_1x, A_2y) \leq h\, M_{12}(x,y)$, $0 \leq h < 1$ whenever $y \in A_1X \cup A_2X$ or $Ty \in \overline{A_1X}$ and $x \in X$.

Let A_1 and S be compatible and A_2 and T be pointwise R-weakly Commuting. If A_1 or S be continuous then all the A_1, S and T have unique common fixed point.

In the present paper we obtain a generalization of the above theorem by replacing the Banach type contractive condition by a more general contravtive condition that employs a contractive gauge fucntion ϕ. The theorem employs the notion of pointwise R-weak commutativity as well as compatibility.

2. Results.

If $\{A_1\}$, $i = 1, 2, 3, \ldots$, S and T be selfmappings of a metric space (X,d), in the sequel for each $i > 1$ we shall denote

$M_{1i}(x,y) = \{d(Sx, Ty), d(A_1x, Sx), d(A_iy, Ty), [d(A_1x, Tx) + d(A_iy, Sx)]/2\}$.

Theorem . Let$\{A_i\}$, $i = 1, 2, 3, \ldots$, S and T be selfmappings of a complete metric space (X, d) such that

(i) given u in X there exists y, v in X such that $A_1u = Ty$, $A_2y = Sv$.

(ii) $d(A_1x, A_iy) < M_{1i}(x,y)$ whenever $M_{1i}(x,y) > 0$,

(iii) $d(A_1x, A_2y) < \phi\, M_{12}(x,y)$,

where $\phi : R_+ \to R_+$ is an upper semicontinuous (to be abbreviated as *u.s. continuous*) function such that $\phi(t) < t$ fro each $t > 0$. Let A_1 and S be compatible and A_2 and T be pointwise R-weakly Commuting. If A_1 or S be continuous then all the A_i, S and T have a unique common fixed point.

Proof. Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule

$$y_{2n} = A_1x_{2n} = Tx_{2n+1}, \qquad y_{2n+1} = A_2x_{2n+1} = Sx_{2n+2}.$$

This can be done by virtue of (i). If $A_1x_{2n} = A_2x_{2n+1}$ or $A_2x_{2n+1} = A_1x_{2n+2}$ for some value of n, it becomes easier to establish the existence of the fixed point. So let us assume that $A_1x_{2n} \neq A_2x_{2n+1}$ and $A_2x_{2n+1} \neq A_1x_{2n+2}$ for every value of n. Then, by virtue of (ii), we obtain

(2.1) $d(y_{2n}, y_{2n+1}) \leq \phi(d(y_{2n-1}, y_{2n})) < d(y_{2n-1}, y_{2n})$,

and

(2.2) $d(y_{2n-1}, y_{2n}) \le \phi(d(y_{2n-2}, y_{2n-1})) < d(y_{2n-2}, y_{2n-1})$.

We thus see that $\{d(y_n, y_{n+1})\}$ is a strictly decreasing sequence of positive numbers and hence tends to a limit $r \ge 0$. Suppose $r > 0$. Then relation (1) on making $n \to \infty$ and in view of *u.s. continuity* of ϕ yields $r \le \phi(r) < r$, a contradiction.

Hence $r = \lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. We show that $\{y_n\}$ is a Cauchy sequence. Suppose it is not. Then there exists an $\epsilon > 0$ and a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $d(y_{n_i}, y_{n_{i+1}}) > 2\varepsilon$. Since $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, there exist integers m_i satisfying $n_i < m_i < n_{i+1}$ such that $d(y_{n_i}, y_{m_i}) \ge \varepsilon$. If not, then

$d(y_{n_i}, y_{n_{i+1}}) \le d(y_{n_i}, y_{n_{i+1}-1}) + d(y_{n_{i+1}-1}, y_{n_{i+1}}) < \varepsilon + d(y_{n_{i+1}-1}, y_{n_{i+1}}) < 2\varepsilon]$

a contradiction. If m_i be the smallest integer such that $d(y_{n_i}, y_{m_i}) \ge \varepsilon$, then

$$\varepsilon \le d(y_{n_i}, y_{m_i}) \le d(y_{n_i}, y_{m_i-2}) + d(y_{m_i-2}, y_{m_i-1}) + d(y_{m_i-1}, y_{m_i})$$
$$< \varepsilon + d(y_{m_i-2}, y_{m_i-1}) + d(y_{m_i-1}, y_{m_i}).$$

That is, there exist integers m_i satisfying $n_i < m_i < n_{i+1}$ such that $d(y_{n_i}, y_{m_i}) \ge \varepsilon$ and

(2.3) $\lim_{n\to\infty} d(y_{n_i}, y_{m_i}) = \varepsilon$.

Without loss of generality we can assume that n_i is odd and m_i even. Now, by virtue of (2.1), we have

$$d(y_{n_i+1}, y_{m_i+1}) \le \phi(\max\{d(y_{n_i}, y_{m_i}), [d(y_{n_i+1}, y_{m_i}) + d(y_{m_i+1}, y_{n_i})]/2\})$$

Now on letting $n_i \to \infty$ and in view of (2.3) and u.s. continuity of ϕ, the above relation yields $\varepsilon \le \phi(\varepsilon) < \varepsilon$, a contradiction. Hence $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists a point z in X such that $y_n \to z$. Also

$$y_{2n} = A_1 x_{2n} = Tx_{2n+1} \to z \text{ and } y_{2n+1} = A_2 x_{2n+1} = Sx_{2n+2} \to z.$$

Suppose S is continuous. Then $SA_1x_{2n} \to Sz$ and $SSX_{2n} \to Sz$. Compatibility of A_1 and S implies $\lim_{n\to\infty} d(A_1x_{2n}, SA_1x_{2n}) = 0$, that is, $A_1Sx_{2n} \to Sz$. We claim that $z = Sz$. If $z \ne Sz$, the inequality

$$d(A_1Sx_{2n}, A_2x_{2n+1}) \le \phi(\max\{d(SSx_{2n}, Tx_{2n+1}), d(A_1Sx_{2n}, SSx_{2n}),$$
$$d(A_2x_{2n+1}, Tx_{2n+1}), [d(A_1x_{2n}, Tx_{2nz+1}) + d(A_2x_{2n+1}, SSx_{2n})]/2\}),$$

on letting $n \to \infty$ leads to $d(z, Sz) \le \phi(d(z, Sz)) < d(z, Sz)$, a contradiction. Therefore $z = Sz$. Similarly, If $z \ne A_1z$ the inequality

$$d(A_1z, A_2x_{2n+1}) \leq \phi\,(max\,\{d(Sz, Tx_{2n+1}), d(A_1z, Sz),$$
$$d(A_2x_{2n+1}, Tx_{2n+1}), [d(A_1z, Tx_{2n+1}) + d(A_2x_{2n+1}, Sz)]/2\}),$$

on making $n \to \infty$ yields, $d(z,A_1z) \leq \phi\,(d(z,A_1z)) < d\,(z, A_1z)$, a contradiction. Thus $z = A_1 z = Sz$. By virtue of (i), there exists a point w in X such that $z = A_1 z = Tw$. We now show that $z = A_2 w = Tw$. If $z \neq A_2 w$ we get

$$d(A_1z, A_2 w) \leq (max\,\{d(Sz, Tw), d\,(A_1z, Sz), d\,(A_2w, Tw), [d\,(A_1z, Tw) + d\,(A_2 w, Sz)]/2\}),$$

i.e. $d\,(z, A_2 w) \leq \phi\,\,(d\,(z, A_2 w)) < d\,(z, A_2 w)$.

a contradiction. Hence $z = A_1z = Sz = Tw = A_2w$.

Pointwise R-weak commutativity of A_2 and T implies that $d\,(A_2Tw, TA_2 w) \leq R\,d\,(A_2w, Tw) = 0$ for some $R > 0$, we get $A_2Tw = TA_2\,w$, i.e. $A_2 z = Tz$. If $z \neq A_2z$, the inequality

$$d(A_1z, A_2 z) \leq \phi\,(max\,\{d(Sz, Tz), d\,(A_1z, Sz), d\,(A_2z, Tz), [d(A_1z, Tz) + d\,(A_2 z,Sz)]/2\}).$$

leads to $d(z, A_2 z) \leq \phi\,(d(z, A_2 z)) < d(z, A_2 z)$ a contradiction. Hence z is a common fixed point of A_1, A_2, S and T. Now if $z \neq A_i z$ for some $i > 2$, (ii) yields

$$d(A_1z, A_i z) \leq max\,\{d(Sz, Tz), d\,(A_1z, Sz), [d\,(A_iz, Tz), [d(A_1z, Tz) + d(A_1 z,Sz)]/2\}).$$

That is, $(d(z, A_i z)) < d(z, A_i z)$ a contradiction. We have thus shown that z is a common fixed point of all the A_i, S and T if S is assumed continuous. The proof is similar when T is assumed continuous.

Next, suppose that A_1 is continuous. Then $A_1 A_1 x_{2n} \to A_1z$ and $A_1Sx_{2n} \to A_1z$. Compatibility of A_1 and S implies $\lim_{n\to\infty} d(A_1 Sx_{2n}, SA_1x_{2n}) = 0$, that is, $SA_1 x_{2n} \to A_1z$. If $z \neq A_1z$ the inequality

$$d\,(A_1 A_1 x_{2n}, A_2 x_{2n+1}) \leq \phi\,(max\,\{d(SA_1 x_{2n}, Tx_{2n+1}), d\,(A_1 A_1 x_{2n}, S A_1 x_{2n}), d\,(A_2\,x_{2n+1}, Tx_{2n+1}), [d\,(A_1 A_1 x_{2n+1}, Tx_{2n+1}) + d\,(A_2 x_{2n+1}, SA_1x_{2n})]/2\}),$$

in view of u.s. continuity of ϕ will imply $d\,(z, A_1 z) \leq \phi\,(d(z, A_1z)) < d(z,A_1z)$. Hence $z = A_1z$. By virtue of (i), there exist points w, u in X such that $z = Tw = A_1z$ and

(2.4) $\qquad A_2 w = Su.$

We now show that $z = Tw = A_2\,w$. If $z \neq A_2w$, the inequality

$$d\,(A_1\,x_{2n}, A_2 w) \leq \phi\,(max\,\{d(S\,x_{2n}, Tw), d\,(A_1\,x_{2n}, S\,x_{2n}), d\,(A_2\,w, Tw), [d\,(A_1\,x_{2n}, Tw) + d\,(A_2\,w, Sx_{2n})]/2\}).$$

on letting $n \to \infty$ yields $d(z, A_2 w) \leq \phi\,(d(z, A_2 w)) < d\,(z, A_2w)$, a contradiction.

Hence $z = Tw = A_2 w$.

Pointwise R-weak commutativity of A_2 and T implies that $d(A_2Tw, TA_2w) \leq Rd(A_2w, Tw)=0$ for some $R > 0$, that is, $A_2Tw = TA_2w$, or $A_2z = Tz$. Now, if $z \neq A_2 z$, the inequality

$$d(A_1 x_{2n}, A_2 z) \leq \phi(\max\{d(Sx_{2n}, Tz), d(A_1 x_{2n}, Sx_{2n}), d(A_2 z, Tz), [d(A_1 x_{2n}, Tz) + d(A_2 z, Sx_{2n})]/2\}).$$

will yield $d(z, A_2 z) \leq \phi(d(z, A_2 z)) < d(z, A_2 z)$. Thus $z = A_1 z = A_2 z = Tz$. Since by (2.4) $A_2 w = Su$ for some $u \in X$, we show that $z = A_1 u = Su$. If not, the inequality

$$d(A_1 u, A_2 z) \leq \phi(\max\{d(Su, Tz), d(A_1 u, Su), d(A_2 z, Tz), [d(A_1 u, Tz) + d(A_2 z, Su)]/2\}),$$

will yield $d(A_1 u, z) \leq \phi(d(A_1 u, z)) < d(A_1 u, z)$, a contradiction. Hence $A_1 u = Su = z$. Since A_1 and S are compatible, we get $A_1 Su = SA_1 u$, that is $A_1 z = Sz$. Hence z is a common fixed point of A_1, A_2, S and T. Finally, if $z \neq A_i z$ for some $i > 2$, we get

$$d(A_1 z, A_i z) < \max\{d(Sz, Tz), d(A_1 z, Sz), d(A_i z, Tz), [d(A_1 z, Tz) + d(A_i z, Sz)]/2\}),$$

that is, $d(z, A_i z) < d(z, A_i z)$, a contradiction. Hence z is a common fixed point of all the A_i, S and T when A_1 is continuous. The proof is similar when A_2 is assumed continuous.

Uniqueness of the common fixed point follows easily. This completes the proof of the theorem.

As a direct consequnce of the above theorem we get the following **Corollary**. Let A_1, A_2, S and T be selfmapping of a complete metric space (X, d) such that

(i) $A_1 X \subset TX$, $A_2 X \subset SX$,

(ii) $d(A_1 x, A_2 y) \leq \phi(\max\{d(Sx, Ty), d(A_1 x, Sx), d(A_2 y, Ty)\})$

$x, y \in X$.

where $\phi : R_+ \to R_+$ is an upper semicontinuous function such that $\phi(t) < t$ for each $T > 0$. Let $A_1 S$ be compatible and A_2, T be pointwise R-weakly communting. If A_1 or S be continuous than A_1, A_2, S and T have a unique common fixed point.

We now give an example of mappings satisfying the conditions of our theorem and having a unique common fixed point.

Example .

Let $X = [2, \infty)$ with usual metric d. Define mappings $A_i, S, T : X \to X$,

$i=1,2,3\ldots$, by

$Sx = x + 5$ when $x > 2$, $Sx = 2$ when $x = 2$

$Tx = 4x$ when $x \geq 3$, $Tx = 2$ when $x < 3$,

$A_1x = 2$ for all x,

$A_2x = 2$ when $x = 2$, $A_2x = 2x$ when $x > 2$,

$A_ix = 2\,(3 + 1/i)$ when $x > 3 + 1/i$

$A_ix = 2$ when $x \leq 3 + 1/i$.

Then$\{A_i\}$, S and T satisfy all the conditions of our theorem and have a unique common fixed point $x = 2$.

In view of this example, a few observation will be in order:

1. **Theorem 5.1.** of Jachymski [4] assumes, for each $i > 2$, $A_i X \subset SX$ and A_i to be compatible with T. These conditions are neither required in our theorem nor are satisfied in the example. It is obvious that $A_i X$ is not contained in SX when $i > 2$, let us consider a decreasing sequence $\{x_n\}$ in $X = [2,\infty)$ such that $x_n \to 3 + (1/i)$. Then $A_ix_n \to 2(3 + (1/i))$ and $Tx_n \to 4(3 + (1/i))$ But $A_iTx_n \to 2(3 + (1/i))$ while $TA_ix_n \to 8(3 + (1/i))$. Hence $lim_n\, d(A_iTx_n, TA_ix_n) \neq 0$, that is, A_i and T are noncompatible.

2. The main theorem of Rhoades et al. [15] requires $A_iX \subset SX \cap TX$ and A_i to be compatible with both S and T for every value of $i > 2$. These conditions are not required in our theorem and are not satisfied in the above example. Moreover, their result requires each A_i, A_j to satisfy an (ε,δ) type contractive condition (the necessary correction in their (ε,δ) condition appeared in Jungck et al [7]). However, in the above example , A_1, A_1, $i>2$, fail to satisfy (ε,δ) condition at $\varepsilon = 4 + (2/i)$.

3. Likewise, the present theorem is more general than Theorem 2 of pant, Joshi and Pande [11] in one respect. In theorem 2 of Pant et al. if we take $\{P_i\} = \{A_1, A_1, A_1, \ldots\}$ and $\{Q_i\} = \{A_2, A_3, A_4, \ldots\}$ then A_1, A_i are required to satisfy an (ϵ, δ) contractive condition for each $i>2$. However, as discussed above, this condition is not satisfied in the above example at $\varepsilon = 4 + (2/i)$.

Remarks. In the above theroem if we let $\phi(t) = ht$, $0 \leq h < 1$, then we obtain the main theorem of Pant et al [12] as special case of the present theorem.

It is clear from the above discussion that our theorem applies to much wider class of mappings than covered by the above mentioned results. Moreover, since other results of this type can be obtained as special cases of the results in [4], [11], [16] mentioned above, they can also be obtained as special cases of our theorem. For a nice comarative study of various results one may refer to Rhoades et al. [15] or Jachymski [4]. Among the results which can be obtained as special cases of the present Theorem, we mention here due to Boyd and Wong [1], Das and Naik [2], Fisher [3], Joshi an Pant [5], Meir and Keeler [8], Pant [9,10], Pant et al [12], Park and Bae [13], Park and Rhoades [14], Rao and Rao [15] and Tivari and Singh [18].

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