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CONTINUITY CONDITION FOR MEASURABLE MULTI-VALUED MAPPING

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1. Introduction. The concept of continuity of a funciton has been the concern of many authors since the time of Newton who defined the concept of limit of real valued function to determine the continuity of a function by showing that if the limit of function exists at a given point and if that limit of the function is equal to the value of function at that point then function is continuous. This concept of continuity of function was generalised from real valued to complex valued function as well as function defined on metric spaces and topological spaces by the use of open sphere and open sets etc. In this generalisation the ontinuity of function is investigated in three main directions viz.

(1) How the continuity of function is affective by changing the natue of a function without being changing the nature of a function without being changing the nature of spaces?

(2) How the continuity of funciton is affected by changing the nature of space under consideration, without changing the nature of function?

(3) How the continuity of function is affected when the nature of function as well as space under consideration, both are subject to change?

Since we are investigating the concept of continuity related to multi-valued maping so we will confine above three aspects. Many authors like Plis, Aumann, Jacob, Himmelberg, have tried to investigate such relationship in the multi-valued mapping under specific condition on a metric space. In the work of all these author's the principle involved has been generalisation of continuity of open sets in metric spaces in terms of measure i.e. any multi-valued function to be continuous, we must show that for any $\varepsilon > 0$ there exist an open set E_ε contain in the given metric space such that the measure $\mu(E_\varepsilon) < \varepsilon$, where μ is Randon measure, when the restriction of measurable mapping to E_ε is continuous. Plis has tried to obtain the continuity condition for a measurable multi-valued mapping by assuming the metric space to be a compact space, while Casting has tried to obtain a similar condition for the continuity of measurable multi-valued mapping when space is seperable instead of compact. Jacob further generalised the condition of both Plis and Casting. His study was moltivated due to the considerable direction porvided by the work of Casting, Kurtawaski, Plis and Michal.

Our study will also confine to such similar situation for multi-valued mapping when it has compact value.

2. Preliminary. Through out the paper (T,e) will denote compact metric space and (X, d) be a seperable metric space. A multifunction $\Omega :\to X$ is a subset of $T\times X$ with domain equal to T. If $B\in X$, then $\Omega^{-1}(B) = \{t \in T \mid \Omega\ (t) \cap B \neq \phi\}$.

Definition 2.1. A multifunction Ω :T$\to X$ is (weakly) measurable if $\Omega^{-1}(B) = \{t \in T \mid \Omega\ (t) \cap B \neq \phi\}$ is measurable for every closed (respectively, open) subset B of X.

Definition 2.2. If $\Omega : T\to X$ is a multifunction, then φ_Ω is the real valued function defined on $T \times X$ by

$$\varphi_\Omega(t, x) = d\ [X, \Omega(t)], \text{ for } (t,x) \in T \times X.$$

Definition 2.3. If X be a seperable metrizable space, then X is said to be *Polish*, if X is seperable and metrizable by a complete metric. Again, if X is metrizable and continuous image of Polish space then X is called *Souslin Space.*

Definition 2.4. For the determination of continuity condition in multi-valued mapping we need following

(a) If $t_0 \in T$, and $S_\varepsilon(t_0) = \{t \in T \mid \rho(t, t_0) < \epsilon\}$ then $\xi(t_0) = \{S_\varepsilon(t_0), \epsilon > 0\}$ is said to be *neighborhood filter base* at t_0.

(b) The *grill* of $\xi(t_0)$ is defined and denoted as

$\xi''(t_0) = \{ S''(t_0) \subset T \mid S''(t_0) \cap S_\varepsilon(t_0) \neq 0, \text{ for } \varepsilon > 0 \}$

(c) If range of Ω is non empty closed subset of X then the *Pseudo limit superior* of Ω as $t\to t_0$ [abbreviated : $p\text{–}limsup_{t\to t_0}\Omega(t)$] is defined to be

$\cap_{S_\varepsilon(t_0)\in\xi(t_0)} CL\ [\cup_{t\in S_\varepsilon(t_0)}\Omega(t)]$

where CL denotes the closure,

and the *Pseudo limit inferior* of Ω as $t \to t_0$ (abbreviated : *p-lim* $inf_{t \to t_0}$) is defined by

$\bigcap_{S''(t_0) \in \xi(t_0)} CL\, [\bigcup_{t \in S''(t_0)} \Omega(t)]$

(d) Ω is *Pseudo upper semicontinuous* at $t_0 \to T$ (abbreviated : *p–usc* at $t_0 \in T$) if *p–lim sup* $_{t \to t_0} \Omega(t) \subset \Omega(t_0)$; Ω is *Pseudo lower semi continuous* at $t_0 \in T$ [abbreviated : *p–lsc* at $t_0 \in T$] if *p-limit*$_{t \to t_0} \Omega(t) \supset \Omega(t_0)$; Ω is Pseudo continuous [abbreviated : *p-continuous* at $t_0 \in T$] if Ω is *p-usc* and *p–lsc* at $t_0 \in T$.

(e) Ω is upper semi continuous at $t_0 \in T$ [abbreviated *usc* at $t_0 \in T$] if for each open G containing (t_0) there is an $S_\varepsilon(t_0) \in \xi(t_0)$ such that

$$t \in S_\varepsilon(t_0) \Rightarrow \Omega(t) \subset G .$$

Ω is *Lower semi continous* at $t_0 \in T$ [abbreviated *lsc* at $t_0 \in T$] if for every open G meeting $\Omega(t_0)$ there is an

$$S_\varepsilon(t_0) \in \xi(t_0) \Rightarrow \Omega(t) \subset G \neq 0.$$

(f) If collection of non-empty closed subset of X have finite topology i.e. X has an open base then Ω is continous if any only if Ω is at $t_0 \in T$ and *lsc* at $t_0 \in T$. The following result is due to Himmerlberg, Jacob, Van Vleck [5].

Definition 2.5. If T be locally compact and σ– compact Hausdorff space, and X be seperable metric space, and $\Omega : T \to X$ a multifunction. Then among the following statement we have relations (i)$\Rightarrow$(ii)$\Rightarrow$(iii).

(i) Ω is weakly measurable :

(ii) $P\,\xi_\Omega(t,x)$ is measurable function of t for each $x \in X$;

(iii) For all $\in > 0$ there exists a closed subset T_ε of T such that $\mu(T - T_\varepsilon) < \in$ and $\xi_\Omega / T_\varepsilon \times X$ is continuous.

The following results are due to Himmelberg [3]:

2.6. Let x be seperable metric space and $\Omega : T \to X$ be multifunction with compact values. Then Ω is mesurable iff. Ω is weakly measurable.

2.7. Let X be seperable metric space and let $\Omega : T \to X$ be a multifunction then

(i) Ω is weakly measurable ,

(ii) $t \to d\,[X, \Omega(t)]$ is measurable function of t for each X,

(iii) $Gr(\bar{F})$ is $A \times B$ – measurable [$\bar{F}$ is defined by $\bar{F}(t) = \overline{F(t)}$]. Then (i)$\Leftrightarrow$ (ii)$\Leftrightarrow$ (iii),

where *Gr* denotes graph.

2.8. Let T be complete, X be Souslin, and $\Omega : T \to X$ a multifunction such that $Gr\ (F)$ is $A \times B$ – measureable then Ω is B– measurable.

3. **Main Result.** Let T be a compact metric space and (X,d) be seperable metric space, $\Omega : T \to X$ be a multifunctioning with compact value and μ be Randon measure, then the following statements are equivalent :

(i) Ω is measurable,

(ii) Ω is weakly measurable ,

(iii) $\xi_\Omega (t,x)$ is measurable function of t for each $x \in X$,

(iv) For all $\in > 0$ there exists a closed subset T_ε of T such that $\mu(T - T_\varepsilon) < \in$ and $\xi_\Omega / T_\varepsilon \times X$ is continuous

(v) For all $\in > 0$ there exists a Souslin space T_ε of T such that $\mu(T - T_\varepsilon) < \in$ and graph $Gr\ (\Omega / T_\varepsilon)$of Ω / T_ε is Souslin sub–space.

(vi) There exist a Souslin space T' of T such that $\mu(T - T') = 0$ and graph $Gr\ (\Omega / T')$ is Souslin.

(vii) For all $\in > 0$ there exists a closed subset T_ε of T such that $\mu(T - T_\varepsilon) < \in$ and Ω / T_ε is *p-usc*.

(viii) For $\in > 0$ there exists a closed subspace T_ε of T such that $\mu(T - T_\varepsilon) < \in$ and $Gr\ (\Omega / T_\varepsilon)$ is closed subst of $T \times X$.

Proof .Our scheme to prove the resutl is

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (i).

and (iv) $\Rightarrow$ (vii) $\Rightarrow$ (viii) $\Rightarrow$ (v).

First Part .

(i) $\Rightarrow$ (ii)

Since the Himmelberg [3] in his paper proved that if T be any arbitrary measurbale space and X be any metric space such that the multi–valued functions $\Omega : T \to X$ has compact value then mesurability and weak measurability of multi–valued function are equivalent. Hence (i) $\Rightarrow$(ii) .

(ii) $\Rightarrow$ (iii)

It follows directly by the virtue of Lemma 3 due to Himmelberg–Jacob–Vanveleck [5].

(iii) $\Rightarrow$ (iv)

By the same said Lemma.

(iv) $\Rightarrow$ (v)

To prove it, since for any $\in > 0$ and $\xi_\Omega | T_\varepsilon \times X$ is continuous, we claim $Gr\ (\Omega / T_\varepsilon)$ is closed. To have it we know $\xi_\Omega : T \times X \rightarrow X$ defined by $\xi_\Omega\ (t,x) = d\ [x, \Omega\ (t)]$ and by given (iv) there exists a closed subset T_ε of T such that $\mu\ (T - T_\varepsilon) < \in$ and $\xi_\Omega | T_\varepsilon \times X$ is continous and therefore if (t_n, x_n) be any arbitrary convergent sequence in $G_r\ (\Omega / T_\varepsilon)$ s.t. $(t_n, x_n) \rightarrow (t, x)$ then by the continuity of $\xi_\Omega / T \times X$, we have,

$$d\ [x, \Omega\ (t)] = \xi_\Omega\ (t,x) = Lim\ d\ [x_n, \Omega\ (t_n)] = 0$$

because for each $t_n \in T_\varepsilon$, $t_n \in \Omega\ (t_n)$ and therefore $d\ [x, \Omega\ (t)] = 0$ but t_ε being closed subset of compact metric space T so it is compact and every compact metric space is complete so (t_n) in T is connvergent to t, so $t \in T_\varepsilon$. Further Ω has compact value therefore $\Omega\ (t)$ is compact subset of X, which is Hausdorff and every compact subset of Hausdorff is closed hence $\Omega\ (t)$ is closed, and

$d\ [x, \Omega\ (t)] = 0$, so $x \in \Omega\ (t) \Rightarrow (x, t) \in Gr\ (\Omega / T_\varepsilon)$ hence $Gr\ (\Omega / T_\varepsilon)$ of Ω / T_ε is closed. Again we know every compact space is seperable therefore T_ε being the closed subset of compact metric space is compact, so it must be seperable. Moreover every compact space is locally compact. So T_ε is locally compact seperable and every locally compact seperable metric space is an open subset of compact metric space, it follows that every space is polish, hence it is Souslin

thus T_ε and $G_r\ (\Omega / T_\varepsilon)$ are Souslin space because they are closed subspace of suslin space T and $T \times X$ respectively. Moreover we know that every closed subspace of Souslin space is Souslin, hence the result.

(v) $\Rightarrow$ (vi)

Let us select $T_{1/n}$ for each $n = 1,2,...$ by taking n so large that $1/n < \in$ and let

$$T' = \cup \{ T_{1/n} / n = 1,2\ ... \}$$

We claim

(a) $\mu(T - T') = 0$ and

(b) Graph $Gr\ (\Omega / T')$ is Souslin.

To prove (a), we know from given (v) that $(T - T_\varepsilon) < \in$.

So we have $\mu(T - T') < \lim\limits_{n \to \infty} 1/n \Rightarrow \mu(T - T') = 0$, but we know that $\mu(T-T') > 0$, hence $\mu(T-T') = 0$.

To prove (b), we have $Gr\ (\Omega\ /T') = \cup_n\ Gr\ (\Omega\ /T_{1/n})$.

and again by (v) we have $Gr\ (\Omega\ /T_{1/n})$ of $(\Omega\ /T_{1/n})$ is a Souslin space of $T \times X$ and countable union of Souslin subspace is Souslin. Hence $Gr\ (\Omega\ /T')$ is Souslin.

Thus the result.

(vi) ⇒ (i)

Now to prove it, it is sufficient to show that inverse image of any closed subset of X under Ω is closed in T. For this, let B be any arbitrary closed subset in X. Then we claim $\Omega^{-1}(B)$ is closed. To have this claim let T_1, T_2, T_3, ..., T_n be any arbitary covering of T where each T_i is a closed subspace, since each closed subspace of compact space T is compact therefore each T_i is compact subspace and has a compact value, therefore $\Omega^{-1}(B)$ is the projecion of T of $Gr\ (\Omega) \cap (T\times B)$ but,

$$Gr\ (\Omega) \cap (T\times B) = \cup\ \{Gr(\Omega) \cap (T\times B) \mid n = 1,2\ ...\}$$

Since T being compact hence every convering is reducible to finite sub–covering therefore,

$$\cup\ \{Gr(\Omega) \cap (T_n\times B) \mid n = 1,2\ ...\} = \cup_{i=1}^{k}\{Gr(\Omega) \cap (T_i\times B)\},$$

but for each i, $Gr\ (\Omega) \cap (T_i\times B)$, can be expressed as the union of sets given by

$$Gr\ (\Omega\ /T')\ \cap\ \{(T_i \cap T')\times B)\} \quad ...\ (1)$$

$$\text{and } Gr\ (\Omega)\ \cap\ \{(T_i - T')\times B)\} \quad ...\ (2)$$

Now (1) is Souslin and is projection on the Souslin by our given hypothesis and therefore measurable and (2) is projection on a set of measure zero and we know that countable union of measerable is measurable hence every finite union of measurable set is measurable as every finite set is countable. Thus $\Omega^{-1}(B)$ is measurable being countable union of measurable sets. Therefore Ω is measurable and hence the result.

Second Part

(iv) ⇒ (vii)

Now to prove this we must show that if (t_n) and (x_n) are sequences in T_ε and X respectively such that $x_n \in \Omega\ (t)$ for all n and such that $t_n \to t$ and $x_n \to x$ then we claim $x \in \Omega\ (t)$. Now to prove this condition, since by (iv) we have that $\xi_\Omega / (T_\varepsilon \times X)$ is continuous and therefore we have by definition of ξ_Ω

$$d[x, \Omega(t)] = \underset{n\to\infty}{Lim}\ d[x_n, \Omega(t_n)] = 0$$

and $t \in T$ because T_ε is closed and since $\Omega(t)$ is compact subset of Hausdorff space X so it must be closed therefore,

$$d[x, \Omega(t)] = 0 \Rightarrow x \in \Omega(t)$$

and thus the conditon.

(vii) ⇒ (viii)

Since by given (vii) we have for any fixed $\in >0$ If we select T_ε satisfying the condition (vii) then

we claim $Gr(\Omega/T_\varepsilon)$ is closed subset of $T \times X$. To prove the claim it is sufficient to show that $Gr(\Omega/T_\varepsilon)$ contains all of its limit points. To have this condition, let $(t,x,)$ be any arbitrary limitpoint of $Gr(\Omega/T_\varepsilon)$ then we know by the property of limit point of set in metric space, that if x is any limit point of subset A of metric space X then there exists a sequence (x_n) in A such that $x_n \to x$ therefore if (t, x) is limit point of $Gr(\Omega/T_\varepsilon)$. Then there exsits a sequence (t_n,x_n) in $Gr(\Omega/T_\varepsilon)$ such that $(t_n, x_n) \to (t,x)$. Since Ω/T_ε is p-$u.s.c.$ therefore $\Rightarrow x \in \Omega/T_\varepsilon$ which implies that $(t,x) \in Gr(\Omega/T_\varepsilon)$. Hence $Gr(\Omega/T_\varepsilon)$ is closed subset of $T \times X$.

(viii) ⇒ (v)

The phenomena is trivial.

The results in Section -2 due to Himmelberg, Jacob and Van Vleck [5] becomes particular case of our main result because every compact value of metric space is closed and complete.

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