Jñānābha, Vol. 29, 1999
(Dedicated to Professor J.N. Kapur on his Platinum Jubilee Celebrations)

INVOLUTIONS AND B-INJECTORS OF *Fi22*

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(Received : November 24, 1998)

ABSTRACT

The aim of this paper is to find a simple approach to study centralizers of 3-transpositions in $F_{i22} = \widetilde{M}(22)$ and using these results to determine the B-Injectors of F_{i22}

1. Introduction. A finite group G is said to be of type F_{i22} if G possesses an involution d such that $H = G_G(d)$ is quasimple with $H/\langle d\rangle \cong U_6(2)$ and d is not weakly closed in H with respect to G. For more information one is referred to [2]. The uniqueness of groups of this type is proved in [2]. Here we give a very simple characterization to the centralizer of involutions (3-transpositions) in F_{i22} , and among other results transvections in $GU_6(2)$ are proved to have 3-transpositions. Most of these results are in [2], [3], [7], but here we use very simple notions of group theory and linear algebra to exhibit these results in a more readable form. The concept of N-injectors is due to B. Fischer who proved that if G is solvable, then N-injectors exist and any two of them are conjugate [8]. N-injectors exist and any two of them are conjugate [8]. N-injectors of finite solvable groups, symmetric groups Sn, alternating groups An and classical groups where studied in [11], [7], [6], [5] and [12].

2. Preliminaries and Notations. Let F denote the group F_{i22}. The group $F_{i22} = \langle D\rangle$ where D is a class of involutions (3-transpositions) with the property, $t_1, t_2 \in D$ implies that $o(t_1t_2) = 1, 2,$or 3 where $o(t_1t_2)$ denotes the order of t_1t_2. So if $t_1 \neq t_2$, then $\langle t_1, t_2\rangle$ is a group of order 4 i.e. $\langle t_1, t_2\rangle = 2^2$ or $\langle d_1, d_2\rangle = S_3$ the symmetric group of degree 3. There are 3 classes of involutions j in F_{i22} with representatives $(i) : j = d \in D$ such that $C_F(d) = 2\, U_6(2)$, (ii) : $j = d_1 d_2 = d_2 d_1$ where d_1, d_2 are uniquely determined by j and if $g \in C_G(j)$, one obtains $d_1^g = d_1$, $d_2^g = d_2$ or $d_1^g = d_2$, $d_2^g = d_1$, so $C_F(d_1) \cap C_F(d_2)$ is a normal subgroup of $C(j)$ of order 2^2. $C_F(d_1) = 2U_6,(2)$, this implies that $C_F(d_1)/\langle d_1\rangle = U_6(2)$ and since $d_2 \in$

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$C_F(d_1)$ one obtains $d_2<d_1>$ is a transvection in $U_6(2)$. This can easily be seen as follows:

Let V be a 6-dimentional vector space over the field $GF(4)$, and let $(|)$ be a unitary form on V, this means a function $(|)$:

$(|): V\times V \rightarrow GF(4)$

$(x, y) \rightarrow (x|y)$ such that the following conditions hold

1. $(x_1+x_2|y_1+y_2) = (x_1|y_1) + (x_1|y_2) + (x_2|y_1) + (x_2,y_2)$
2. $(\lambda x|y) = \lambda(x|y)$
3. $(x|\lambda y) = \bar{\lambda}(x|y)$ where $\bar{\lambda}$ is the complex conjugate of λ in $GF(4)$.
4. $(x,y) = \overline{(y,x)}$, se the general unitary group $GU_6(2)$ is defined by $GU_6(2) = \{g \in GL(V):(x^g|y^g) = (x,y)\ \forall\ x,y \in V\}$ and the special unitary group $SU_6(2) = \{g \in GU_6(2): det(g) = 1\}$, the center of $GU_6(2)$ $= Z(GU_6(2)) = \{\lambda I: \lambda\bar{\lambda} = 1\} = \{\lambda I: \lambda^3 = 1\} \leq SU_6(2)$, so $U_6(2) = SU_6(2)/Z(G(U_6(2))$.

Transvections in the unitary group is defined as follows. Let $v \in V$ with $(v|v) = 0$. Define a transvection $t_v: V \rightarrow V$ by $t_v(x) = x - (x|v)v$, sometimes $t_v(x)$ is written as x^{t_v}. This transvection preserves the unitary form for

$(x^{t_v}|y^{t_v}) = (x-(x|v)v|y-(y|v)v = (x,y) - \overline{(y|v)}(x|v) - (x|v)(v|y) + (x|v)$
$\overline{(y|v)}(v|v) = (x,y) - 2(v|y)(x/v) + (x|v)\overline{(y|v)}.0 = (x|y)$.

It is very clear that transvections have 3-transpositions, this means that if t_v, t_w are two transvections then $o(t_v, t_w) = 1,2$, or 3 and $t_v^2=1$ and $t_v\ t_w = t_w\ t_v$ iff $(v/w)=0$.

Lemma1. $C(d_1d_2)$ is of the form 2-group. $U_4(2).2$

Proof.

$C_F(d_1) \cap C_F(d_2) = C_{C_F(d_1)}(d_2)$, So

$(C_F(d_1) \cap C_F(d_2))/<d_1> = C_{U_6(2)}(d_2<d_1>/<d_1>) = C_{U_6(2)}(\bar{d})$

where $\bar{d} = d_2<d_1>/<d_1>$ is a transvection in $U_6(2)$

So $C_{U_6(2)}(\bar{d}_2) = H.U_4(2)$, H is some elementrary abelian 2-group and hence, $\{C(d_1) \cap C(d_2)\}.2 = C(d_1d_2)$. Since $d_1 \in C_{C(d_1)}(d_2)$ one has,

$C(d_1) \cap C(d_2) = 2.\ C_{C(d_1)}(d_2)/<d_1> = C_{U_6(2)} = 2^{1+8}: U_4(2)$ a stabilizer of transvection. Therefore

$C(d_1d_2) = 2.2^{1+8}: U_4(2).2$

3. The third class of involutions in F_{i22} is of the form

$j = j_3 = d_1 d_2 d_3$ where $d_i d_j = d_j d_i$.

Lemma 2. $C_G(d_1 d_2 d_3)$ $2-$group A_6.

Proof. There exist exactly 3 other transpositions d'_1, d'_2, d'_3 and $j= d'_1 d'_2 d'_3$ and $<d_1, d_2, d_3, d'_1, d'_2, d'_3> = 2^5$ for, j can be writtin in exactly two ways as products of 3 commuting transpositions, it follows that $g \in C(j)$ implies that :

(a) $\{d_1, d_2, d_3\}^g = \{d_1, d_2, d_3\}$, $\{d'_1, d'_2, d'_3\}^g = \{d'_1, d'_2, d'_3\}$

or (b) $\{d_1, d_2, d_3\}^g = \{d'_1, d'_2, d'_3\}$.

In any case g normalizes $<d_1, d_2, d_3, d'_1, d'_2\ d'_3>$. So

$C_G(j) \leq N_G(<d_1, d_2, d_3, d'_1, d'_2 d'_3>)$ = stablizer of a hexad in $2^{11}. M_{22} = (2^{11}. M_{22})_{hexad} = 2-group. A_6$.

Now we come to the following conclusion :

Let $C_G(j_1) = 2\ U_6 = H_1$, $C_G(j_2) = 2$- group. $U_4(2).2 = H_2$ and $C_G(j_3) = H_3$.The generalized Fitting group $F^*(G)$ is defined by $F^*(G) = F(G)\ E(G)$ where $F(G)$ is the Fitting group of G and $E(G)$ is a subgroup of G.A. group L is called quasi simple iff $L' = L$ where L' is the derived group of L and $L'/Z(L)$ is simple. So one has :

$F^*(H_2) = O_2(H_2)$ and $F^*(H_3) = O_2(H_3)$ because $F^*(H) = F(H)$.

This can be prooved as follows:

Lemma 3. Let $H = M.A_6$ where M is a 2-group, then $F^*(H) = F(H)$.

Proof. Let $F^*(H) \neq F(H)$ this means that $E(H) \neq \{1\}$ so there exists a quasi simple subnormal subgroup L such that $[F(H), L] = \{1\}$ (use theorem1) and the fact that $L \leq E(H)$. It follows that $L \leq C_H(F(H)$. Since $M < H$, $M \leq O_2(H) = F(H)$ then $ML \leq H$ and

$ML/M \cong ML/M \leq H/M \cong A_6$.In particular $5 \mid L \mid$ as $5 \mid IA_6 \mid$, take $g \in L$ such that order $(g) = 5$, since $[M,L] = 1$. This implies that $[M, g] = 1$ and one obtains $M \leq C_H(g) \leq C_G(g)$. So $|M| \leq |C(g)|$ which is a contradiciton because H does not contain an element g of order 5 with $|M| \leq |C(g)|$. Thus $F^*(H) = F(H)$.

3. Definitions. Let G be a finite group. Define

$d_2(G) = max\ \{| x | : x \leq G;\ x$ is nilpotent of class $\leq 2\}$

$\sigma_2(G) = \{x \leq G;\ x$ is nilpotent of class ≤ 2 and $| x | = d_2(G)\}$

B-Injector of $G = \{H \leq G : H$ is a maximal nilpotent in G and contains a subgroup $x \in \sigma_2(G)\}$.

For these defintions one is referred to [2]. Now we define

$m_k(G)$, $Om_k(G)$ as follows :

$m_k(G) = max\ \{|C_G(x)| : x \in G,\ o(x) = p_1.p_2 \dots p_k,\ p_i \neq p_j$ if $i \neq j$ and let $|C_G(g)| = p_1^{a1} . p_2^{a1} \dots p_k^{ak} .m$ where $p_i \mid m$, $i = 1, \dots, k$ then define $Om_k(G) = max\ \{|C_G(x)| : o(x) = p_1.p_2, \dots p_k,\ 2 < p_1 < p_2 < \dots < p_k\}$. So we get the following critirion : If $H \leq G$ such that H is nilpotent and H has at least k prime divisors different from 2 then $|H| \leq O\,m_k(G) \leq m_k(G)$. The proof goes as in [1].

Theorem 2. [1] If $F^*(G) = O_p(G)\,Z(G)$ then B-injectors of G are of the form $PZ(G)$ where P is a p-Sylow subgroup, where $O_p(G)$ is the unique maximal normal p-group G and $Z(G)$ is the centre of G.

Corollary . G is N-constrained iff $E(G) = \{1\}$.

Theorem 2. Let B be a B-injector in F_{i22}, it follows that if $2 \mid |B|$ then B is a 2-Sylow subgroup.

Proof. If $2 \mid |B|$ then $2 \mid |Z(B)|$, so there exists an involution z such that $B \leq C(Z)$ and z is conjugate to j_1 or j_2 or j_3. If z is conjugate to j_2 or z is conjugate to j_3 then there exists a subgroup H such that $B \leq H \leq G$ and $F^*(G) = O_2(H)$. So the claim follows from theorem 1.

If z is conjugate to j_1, then with out loss of generality, $B \leq C(j_1) = 2U_6(2)$. So the claim follows from results on classical groups [12]. Assume that B is a B-injector and $2 \nmid |B|$ then B is a p-group for: Since $|B| \geq 2^{11}$ and $m_3(F_{i22}) = 30 < 2^{11}$ it follows that $|B|$ has at most 2 prime divisors. If B has exactly two prime divisors this implies that $|B| \geq O\,m_2(F_{i22}) = 21$ which is a contradiction. Therefore B is a p- group, and since $|B| \geq 2^{11}$, this means that B is a 3-Sylow subgroup. Concequently the B-injectors are either 2-Sylow subgroups or 3-Sylow subgroups. Assume that the B-injectors are 3-Sylow sub groups, this implies that there exists a 3-subgroup of order $\geq 2^{11}$ and class ≤ 2. Since the 3-Sylow subgroups of F_{i22} are isomorphic to the 3-Sylow subgroups of $O(7,3)$, it follows that there exists a 3-subgroup $Y \leq O(7,3)$ of order $\geq 2^{11}$ and class ≤ 2, which is impossible. Therefore the B-injectors of F_{i22} are 2-Sylow subgroups and this completes the proof of the theorem.

Achnowledgement

My thanks are due to Prof. J. Schäffer for his helpful discussions.

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