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# ON COMMON FIXED POINTS OF ASYMPTOTICALLY REGULAR MAPPINGS

By

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## ABSTRACT

Defining a new class of mappings, an attempt is made for proving more general fixed points theorems in metric space. Our results extend and generalize essentially known results of Das and Naik, Fisher, Jungck, Rhoades, Sessa, Khan and Swaleh, Singh and Tiwari and several others.

**1. Introduction.** Rhoades, Sessa, Khan and Swaleh [13] proved following theorem:

**Theorem 1.** Let $A$, $S$ and $T$ be three selfmaps of a complete metric space $(X, d)$ satisfying

$$d(Sx, Ty) \leq a_1 d(Sx, Ax) + a_2 d(Tx, Ax) + a_3 d(Sy,Ay) + a_4 d(Ty,Ay) + a_5 d(Sx,Ay) + a_6 d(Tx,Ay) + a_7 d(Sy,Ax) + a_8 d(Ty,Ax) + a_9 d(Ax,Ay). \quad ...(1.1)$$

for all $x$, $y$ in $X$ where $a_i = a_i(x,y)$, $i = 1, 2, ..., 9$, are nonnegative funcitons such that

$$\max\{\sup_{x,y\in X}(a_1+a_2+a_5+a_6), \sup_{x,y\in X}(a_3+a_4+a_7+a_8), \sup_{x,y\in X}(a_5+a_6+a_7+a_8+a_9)\} < 1 \quad (1.2)$$

A weakly commutes with $S$ and $T$, (1.3)

there exists a sequence which is asymptotically $S$-regular and $T$-regular with respect to $A$, (1.4)

$A$ is continuous (1.5)

then $A$, $S$ and $T$ have a unique common fixed point.

The authors also gave an example to show that the condition of continuity of the mapping $A$ can not be dropped. But our claim is that the continuity of $A$ is not required on the whole space. Hence, continuity of $A$ can be replaced by asymptotic continuity which we have to define like orbital continuity due to Ciric [2].

**Definition 1.** (appears in [12]) Let $A$ and $S$ be two selfmaps of $X$ and $\{x_n\}$ a sequence in $X$. Then $\{x_n\}$ is said to be asymptotically $S$-regular with respect to $A$ if

$d\,(Ax_n, Sx_n) \to 0$ as $n \to \infty$.

If $A$ is the identity map of $X$, definition 1 becomes that of Engl [4].

Now we define.

(2) $0\,(A, x_n) = \{Ax_1, Ax_2, \ldots, Ax_n, \ldots\}$ is called asymptotic orbit of $A$, where $\{x_n\}$ is a sequence which is asymptotically $S$-regular with respect to $A$.

(3) $A$ is said to be asymptotically continuous mapping if it is continuous on closure of $0\,(A, x_n)$, denoted by $\overline{0\,(A, x_n)}$.

Clearly continuous mapping is asymptotically continuous, but converse is not necessarily true. [See example 1].

(4) $X$ is said to be $A$-asymptotically complete if every Cauchy sequence of the form $\{Ax_n\}$ converges in $X$.

(5) Sessa [14] : Two selfmaps $A$ and $S$ of $X$ are said to be weakly commuting mappings if $d(SAx, ASx) \le d(Ax, Sx)$ for all $x$ in $X$.

Now we present our main results.

## 2. Main Results

**Theorem 2.** Let $A$, $S$ and $T$ be three selfmaps of a metric space $(X, d)$ satisfying (1.1), (1.2), (1.3), (1.4) and

$A$ is asymptotically continuous and $X$ is $A$-asymptotically complete.. (2.1)

Then $A$, $S$ and $T$ have a unique common fixed point.

**Proof.** Let $\{x_n\}$ be a sequence satisfying (1.4). Using (1.1),

$$\begin{aligned} d(Ax_n, Ax_m) &\le d(Ax_n, Sx_n) + d(Sx_n, Tx_m) + d(Tx_m, Tx_m) \\ &\le d(Ax_n, Sx_n) + a_1 d(Sx_n, Ax_n) + a_2 d(Tx_n, Ax_n) + a_3 d(Sx_m, Ax_m) \\ &\quad + a_4 d(Tx_m, Ax_m) + a_5 d(Sx_n, Ax_m) + a_6 d(Tx_n, Ax_m) + a_7 d(Sx_m, Ax_n) \\ &\quad + a_8 d(Tx_m, Ax_n) + a_9 d(Ax_n, Ax_m) + d(Tx_m, Ax_m) \end{aligned}$$

where $a_i = a_i\,(x_n, x_m)$. With the use of triangle inequality, rearranging the terms and then taking the *Lim Sup* of both sides, we get $\{Ax_n\}$ is Cauchy sequence. Since $X$ is asymptotically complete, Let $z = Lim\, Ax_n$. We also have, $Sx_n \to z$ and $Tx_n \to z$. Also using (2.1), $A^2x_n \to Az$, $ASx_n \to Az$ and $ATx_n \to Az$. Now

$d(SAx_n, Az) \le d(SAx_n, ASx_n) + d(ASx_n, Az)$ whence $SAx_n \to Az$. Similarly from (1.3) $TAx_n \to Az$ further, from (1.1) with $a_i = a_i(Ax_n, z)$.

$$\begin{aligned} d(Az, Tz) &\le d(Az, SAx_n) + d(SAx_n, Tz) \\ &\le d(Az, SAx_n) + a_1 d(SAx_n, A^2x_n) + a_2 d(TAx_n, A^2x_n) + a_3 d(Sz, Az) \\ &\quad + a_4 d(Tz, Az) + a_5 d(SAx_n, Az) + a_6 d(TAx_n, Az) + a_7 d(Sz, A^2x_n) \\ &\quad + a_8 d(Tz, A^2x_n) + a_9\, d(A^2x_n, Az) \\ &\le d(Az, SAx_n) + a_1 d(SAx_n, A^2x_n) + a_2 d(TAx_n, A^2x_n) \\ &\quad + (a_3+a_4+a_7+a_8)\, max\{d(Az, Sz), d(Az, Tz)\} \\ &\quad + (a_5+a_6+a_7+a_8+a_9)\, max\, \{d(SAx_n, Az), d(TAx_n, Az), d(A^2x_n, Az)\}. \end{aligned}$$

Taking the *lim Sup,* we have

$$d(Az, Sz) \le \underset{x,y\, \varepsilon X}{Sup} (a_3 + a_4 + a_7 + a_8)\, max\, \{d(Az, Sz), d(Az, Tz)\}$$

Similarly

$$d(Az, Tz) \le \underset{x,y\, \varepsilon X}{Sup} (a_1 + a_2 + a_5 + a_6)\, max\, \{d(Az, Sz), d(Az, Tz)\}$$

Then from (1.2) it follows $Az = Sz = Tz$.

Again using (1.1) with $a_i = a_i(x_n, Ax_n)$,

$$\begin{aligned} d(Sx_n, TAx_n) \le\; & a_1 d(Sx_n, Ax_n) + a_2\, d(Tx_n, Ax_n) + a_3 d(SAx_n, A^2x_n) \\ & + a_4\, d(TAx_n, A^2x_n) + (a_5+a_6+a_7+a_8+a_9)\, max\, \{d(SAx_n, A^2x_n), \\ & d(Tx_n, A^2x_n), d(SAx_n, Ax_n), d(TAx_n, Ax_n), d(Ax_n, A^2x_n)\}. \end{aligned}$$

Taking *Lim Sup* of both sides, yields

$$d(z, Az) \le \underset{x,y\, \varepsilon X}{Sup} (a_5 + a_6 + a_7 + a_8 + a_9)\, d(z, Az) < d(z, Az),$$

which gives $z = Az$ and hence $z$ is a common fixed point of $A$, $S$ and $T$. Uniqueness follows easily. This completes the proof.

Replacing the asymptotic continuity of $A$ with asymptotic continuity of $S$ or $T$, we have the following theorem.

**Theorem 3.** Let $A$, $S$ and $T$ be three selfmaps of a metric space $(X,d)$ satisfying condition (1.1), (1.4), $a_1 + a_2 + a_3 + a_4$ bounded on $X$ and $\underset{x,y\, \varepsilon X}{Sup} (a_5 + a_6 + a_7 + a_8 + a_9) < 1$. If $T$ is asymptotically continuous, weakly commuting with $A$ and $S$; and $X$ is $A$-asymptotically complete, then $T$ has a fixed point.

An analogous theorem can be proved using asymptotic continuity of $S$ instead of $T$. Note that in general, $A$, $S$ and $T$ not necessarily have a common fixed point (Example 3).

**Theorem 4.** Let $\{S_n\}$ be a sequence of selfmaps of a metrica space $(X, d)$ and $A$, a selfmap satisfying with $i \neq j$,

$$d(S_i x, S_j y) \leq a_1 d(S_i x, Ax) + a_2 d(S_j x, Ax) + a_3 d(S_i y, Ay) + a_4 d(S_j y, Ay) + a_5 d(S_i x, Ay) + a_6 d(S_j x, Ay) + a_7 d(S_i y, Ax) + a_8 d(S_j y, Ax) + a_9 d(Ax, Ay)$$

for all $x, y$ in $X$, where $a_k = a_k(x,y)$, $k = 1,2,...,9$, are nonnegative functions satisfying (1.2). If $A$ weakly commutes with each $S_n$ and there exists an asymptotically $S_n$-regular sequence with respect to $A$ for every $n = 1,2,...$, such that $A$ is asymptotically continuous and $X$ is $A$-asymptotically complete. Then the family $\{A, S_1, S_2, ...\}$ has a uniqeue common fixed point.

**Examples.**

In support of theorem 2, we give following example:

(1) Let $X = [0,1)$ with usual metric $d$. Let $S = T$ and $A$, the selfmaps of $X$ defined by

$S(x) = x/(x+16)$ if $x \,\varepsilon\, [0, 1/2]$ and $Ax = x/8$ if $x \,\varepsilon\, [0, 1/2]$

$= 1/8$ if $x \,\varepsilon\, (1/2, 1)$ $= 1/4$ if $x \,\varepsilon\, [1/2, 1]$

Clearly $A$ and $S$ weakly commuting mappings. Let $x_n = 1/(n+2)$ (or any sequence $x_n \neq 0$, converging to $0$), since $d(Sx_n, Ax_n) = x_n(x_n+8)(x_n+16)/8$, $\{x_n\}$is asymptotically $S$-regular w.r.t. $A$. Inequality (1.1) also holds for $a_9 = 1/2$, $a_i = 0$, for $i = 1,2,...,8$. $X$ is asymptotically complete. Then all the assumptions of theorem 2 are satisfied of course assumptions of theorem 1 are not satisfied since $A$ is discontinuous and $X$ is not a complete metric space.

The idea of following example appears in [13],

(2) This example shows that conditions of asymptotic continuity of $A$, can not be dropped. Let $x = [0, 1]$ and $S = T$, $A: X \to X$ given by

$Sx = 1/2$ if $x = 0$ and $Ax = 1$ if $x = 0$

$= x/4$ if $x \neq 0$ $= x/2$ if $x \neq 0$

then except asymptotic continuity of $A$, all other assumptions of theorem 2 are satisfied with $a_9 = 1/2$, $a_i = 0$, for $i = 1,2,..., 8$. On the other hand $A$ and $S$ have no fixed point.

(3) Under the assumptions of theorem 3, $A$, $S$ and $T$ are not necessarily have a common fixed point.

Let $X = [0, 1]$, A, S and $T: X \to X$ given by

$Ax = 1$ if $x = 0$ and $Sx = 1/4$ if $x = 0$ $Tx = x/2$ for all $x$ in $X$

$= x$ if $x \neq 0$ $= x/2$ if $x \neq 0$

Let $a_1 = a_9 = 1/2$ and $a_i = 0$ for $i = 2, 3,...,8$.

Then all the assumption of theorem 3 are satisfied, but $0$ is not a fixed point of either $S$ or $A$.

**Remarks.**

(1) Assuming continuity of $A$ in place of asymptotic continuity and $X$ is complete in place of asymptotic completeness, theroms 2,3 and 4 are the theorems 1,1.1 and 4.2 of [13].

(2) Theorem 2 may be regarded as an extension of the well known result of Hardy and Rogers [7], which considered the following condition :

$d\{Tx,Ty\} \leq b_1 d(Tx, x) + b_2 d(Ty,y) + b_3 d(Tx,y) + b_4 d(Ty,x) + b_5 d(x,y)$

for all $x$, $y$ in $X$ where $b_1 \geq 0$, $i = 1, 2,...5$, $b_1+b_2+b_3+b_4+b_5 <1$

(3) Above contractive condition has been also used by Guay and Singh [6] for $max\ \{b_3+b_4+b_5+b_1+b_3\} < 1$.

(4) In Jungck [8], the continuity of the mapping $S = T$ is a consequence of his contractive condition and it is used in his proof. But in theorem 2, the continuity of his mappings $S$ and $T$ are neither assumed nor implied by the contractive condition (1.1).

(5) Das and Naik [3], generalize Jungck's theorem by considering the following condition for commuting mappings,

$d(Sx, Sy) \leq c\ max\{d(Sx,Ax), d(Sy, Ay), d(Sx, Ay), d(Sy,Ax), d(Ax,Ay)\}$

for all $x$, $y$ in $X$ where $0 \leq c \leq 1$. As indicated in Massa [9], this contractive condition is equivalent to the following condition,

$d(Sx, Sy) \leq a_1 d(Sx,Ax)+a_2 d(Sy, Ay)+a_3 d(Sx, Ay)+a_4 d(Sy,Ax)+ a_5 d(Ax,Ay)\}$

for all $x, y$ in $X$, $a_i = a_i(x, y)$, $i = 1,2,...,5$ and $\sup_{x,y\ \varepsilon X} (a_1+a_2+a_3+a_4+a_5)<1$.

Clearly theorem 2 is a generalization of result of Das and Naik, which has been extended also by Fisher [5], Rhoades [11] and several others. We refer [13], for further details.

(6) Theorem 4 can be regarded as an improvement of theorem 1 of Singh and Tiwari [14].

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