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RECIPROCAL CONTINUITY AND FIXED POINT

By

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ABSTRACT

The aim of this paper is to obtain common fixed point theorems for reciprocally continuous mappings. We studied some contractive definitions which are strong enough to generate a common fixed point, but which does not force the mappings to be continuous at the common fixed points. These theorems will thus provide a common fixed point when all the mappinsgs may be discontinuous.

1. Introduction. In recent years a number of interesting results concerning four mappings or a sequence of mappings have been obtained by various authors. The common fixed point theorems for four mappings require each mapping to be compatible or weakly commuting with one or more mappings. Similarly, the theorems concerning sequences of mappings generally require each mapping to satisfy a compatibility condition, a condition on its range and a strong type continuity condition. The most generally common fixed point theorems concerning sequences of mappings are those due to Jachymaski [2] and Pant [7].

In 1986 Jungck [3] introduced the notion of compatibility.

If (X,d) be a metric space, two self maps A and S of X are called compatible if $Lim_n\, d(ASx_n, SAx_n) = 0$ when $\{x_n\}$ is a sequence such that $lim_n Ax_n = lim_n Sx_n = t$ for some t in X.

Two selfmaps A and S are said to be R-weakly commuting [6] at a point x in X if $d\,(ASx, SAx) \leq R\, d(Ax, Sx)$ for some $R > 0$. It is obvious that A and S can fail to pointwise R-weakly only if there is some x in X such that $Ax = Sx$ but $ASx \neq SAx$.

We discuss problem when there exists a contractive definition which is strong enough to generate a fixed point but which does not force the maps to be continuous is still problem [5]. It may be observed in this context that due to Kannan [5] there exist maps which have a discontinuity in their domain but they have fixed points. However, in all the cases maps involved were continous at the fixed point. However, in all the cases maps involved were continuous at the fixed points. In the context of common fixed point op pairs of mappings there is an approach to deal with this problem. This approach is to weaken the notion of continuity and define a condition which is applicable to discontinuous mappings also.

This approach has been adopted by Pant ([6], [7]) in his recent studies, and he introduced the notion of reciprocally continuous mappings.

Two self maps A and S of a metric space (X,d) is called reciprocally continuous if $lim_n\ ASx_n = At$ and $lim_n\ SAx_n = St$ whenever $\{x_n\}$ is a sequence such that $lim_n\ Ax_n = lim_n\ Sx_n = t$ for some t in X.

In the setting of common fixed point theorems for compatible mappings satisfying contractive conditions, continuity of one of the mappings A and S implies their reciprocal continuity but not conversely. We now prove common fixed point theorems for four mappings A,B,S and T for the purpose of this theorem we have the main result

$$M(x,y) = max\ \{d(Sx, Ty), d(Ax, Sx), d(By, Ty)\}.$$

Theorem 1 : Let (A,S) and (B,T) be pointwise R-weakly commuting pairs of selfmappings of a complete metric space (X,d) such that

(i) $AX \subset TX,\ BX \subset SX$

(ii) $d(Ax,By) \le h\ M(x,y),\ 0 \le h < 1,\ x,y \in X.$

Suppose that one of (A,S) and (B,T) be a compatible pair of reciprocally continuous mappings. Then A,B,S and T have a unique common fixed point.

Proof. Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule

$$y_{2n} = Ax_{2n} = Tx_{2n+1}, \qquad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \tag{1}$$

This can be done by virtue of (i). Then using (ii) we obtain

$d(y_{2n}, y_{2n-1}) \le hd(y_{2n-1}, y_{2n}),\ d(y_{2n-1}, y_{2n}) \le hd(y_{2n-2}, y_{2n-1})$

that is, $d(y_n, y_{n+1}) \le hd(y_{n-1}, y_n) \le \dots \le h^n d(y_0, y_1)$

Moreover, for any integer $p > 0$, we get

$$\begin{aligned} d(y_n, y_{n+p}) &\le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \dots + d(y_{n+p+1}, y_{n+p}) \\ &\le (1+h+h^2+\dots+h^{p-1})\ d(y_n, y_{n+1}) \\ &\le (1/(1-h))\ h^n d(y_0, y_1) \end{aligned}$$

This means that $d(y_n, y_{n+p}) \to 0$ as $n \to \infty$. Therefore, $\{y_n\}$ is a Cauchy sequence.

Since X is complete, there exists a point z in X such that $y_n \to z$. Also $y_{2n} = Ax_{2n} = Tx_{2n+1} \to z$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \to z$. (2)
Suppose A and S are compatible and reciprocally continuous mappings. Then, reciprocal continuity of A and S implies that $ASx_{2n} \to Az$ and $SAx_{2n} \to Sz$. Compatibility of A and S yields $lim_{n\to\infty} d(ASx_{2n}, SAx_{2n}) = 0$, that is, $d(Az,Sz) = 0$. Hence $Az = Sz$. (3)
Since $AX \subset TX$, there exists a point w in X such that $Az = Tw$. Using (ii) we now get

$d(Az, Bw) \leq h(M(z,w))$.

$= h(max\{d(Sz, Tw), d(Az, Sz), d(Bw, Tw)\})$,

$= h(d(Az, Bw))$,

that is $Az = Bw$. Thus $Sz = Az = Tw = Bw$. Compatibility of A and S implies that $ASz = SAz$ and, therefore, $AAz = ASz = SAz = SSz$. Similarly, pointwise *R-weak* commutativity of B and T implies that there exists $R > 0$ such that $d(BTw, TBw) \leq R\ d(Bw, Tw) = 0$, that is, $BTw = TBw$ and $BBw = BTw = TBw = TTw$. Using (ii) we get

$d(Az, AAz) = d(Bw, AAz) = d(AAz, Bw) \leq h\ M(Az, w)$

$< h(d\{(SAz, Tw), d(SAz, AAz)\ d(Bw, Tw)\})$,

$d(Az, AAz) \leq hd(Az, AAz)$,

$(1-h)\ d(Az, AAz) \leq 0$,. Hence $Az = AAz$ and $Az = AAz = SA\,z$. Thus Az is a common fixed point of A and S. Similarly, by using (ii) we obtain that Bw is a common fixed point of B and T. But $Az = Bw$. Hence Az is a unique common fixed point of $A,S,B,$ and T. The proof is similar when B and T are assumed compatible and reciprocally continuous. This completes the proof of the theorem.

If $\{A_i\}$, $i = 1,2,3,\ldots,S$ and T be selfmappings of a metric space (X,d) in the sequel for each $i > 1$. We shall denote

$M_{1i}(x,y) = max\{d(Sx, Ty), d(A_1x,Sx), d(A_iy, Ty)\}$.

Corollary 1.1 . Let $\{A_i\}$, $i = 1,2,3, \ldots ,S$ and T be self mappings of a complete metric space (X,d) such that

(i)$'$ $A_1X \subset Tx$ and $A_2X \subset SX$

(ii)$'$ $d(A_1x, A_2y) \leq h(max\{d(Sx, Ty), d(A_1x, Sx), d(A_2y, Ty)\}$, $0 \leq h < 1$,

(iii)$'$ $d(A_1x, A_iy) < max\{d(Sx, Ty), d(A_1x, Sx), d(A_iy, Ty)\}$,

Let S be compatible with A_1 and T be compatible with A_2. If the mappings in one of the compatible pairs (A_1,S) or (A_2,T) be reciprocally continuous then all the A_i, S and T have a unique common fixed point.

Proof. In theorem 5.1 we have discussed $\{x_n\}$ and $\{y_n\}$ in X are Cauchy sequences. Using the same argument in Theorem 5.1, $Sz = Az = Tw = Bw$. Now if $A_1z \leq A_iw$ for some $i > z$, (iii)$'$ yields

$d(A_1z, A_iw) < max\ (d(Sz, Tw), d(A_1z, Sz), d(A_iw, Tw)) = d\ (A_iw, A_1z)$,

a contradiction. Hence $A_1z = A_iw = Tw$. Compatibility of A_1 and S implies that $A_1Sz = SA_1z$ and, therefore, $A_1A_1z = A_1Sz = SA_1z = SSz$. Similarly, compatibility of A_2 and T implies that $A_2A_2w = A_2Tw = TA_2w = TTw$. If $A_1z \neq A_1A_1z$, Using (iii)$'$ we get

$$d(A_1z, A_1A_1z) = d\ (A_1A_1z, A_iw) < max\ (d(SA_1z, Tw)\ d(A_1A_1z, SA_1z), d(A_iw, Tw))$$
$$= d(A_1A_1z, A_iw),$$

a contradiction. Hence $A_1A_1z = A_1Sz = SA_1z = SSz = A_1z$, that is A_1z is a common fixed point of A_1 and S. Similarly A_2w is a common fixed point of A_2 and T. Hence $A_1z = A_2\ w$ is a common fixed point of A_1, A_2, S and T.Moreover, if $A_2w \neq A_iA_2w$ for some $i > 2$, using (iii)$'$ we have

$$d\ (A_1z, A_iA_2w) < d(A_1z, A_iA_2w),$$

a contradiction. Hence $A_1z = A_2w$ is a common fixed point of all $\{A_i\}$, S and T. Uniqueness of the common fixed point follows easily. The proof is similar when A_2 and T assumed reciprocally continuous.

We now give an example to illustrate Theorem 1.

Example 1.1 . Let $X = [2,20]$ and d be the usual metric on X. Define A,B,S and $T : X \to X$ by

$A2 = 2,$ $\qquad$ $Ax = 3\ if\ x > 2$

$S2 = 2,$ $\qquad$ $Sx = 6\ if\ x > 2$

$Bx = 2\ if\ x = 2\ or > 5,$ $\qquad$ $Bx = 6\ if\ 2 < x \leq 5$

$T2 = 2,\ Tx = 12\ if\ 2 < x \leq 5,\ Tx = x-3\ if\ x > 5$.

Then A, B, S and T satisfy all the conditions of Theorem 1 and have a unique common fixed point $x = 2$. It may be noted in this example that A and S are reciprocally continuous compatible maps. But neither A nor S is continuous, not even at their common fixed point $x = 2$. The mappings B and T are *R-weakly* commuting since they commute at their coincidence points. The ranges of A,B,S and T are complete subspaces of X. It is obvious in this example that all the mappings are discontinuous at the common fixed point $x = 2$.

In the next theorem we obtain a generalization of the above theorem replacing the Banach type contractive condition by a more general contractive condition that employs a contractive gauge function ϕ.

If A, B, S and T be self mappings of a metric space (X,d) we shall denote

$$M(x,y) = max\,\{d(Sx, Ty), d(Ax,Sx), d(By, Ty)\}.$$

Also, let $\phi : R_+ \to R_+$ denote an upper semicontinuous function such that $\phi(t) < t$ for each $t > 0$.

Theorem 2. Let (A,S) and (B,T) be pointwise *R-weakly* commuting pairs of self mappings of a complete metric space (X,d) such that

(i) $AX \subset TX$, $BX \subset SX$

(ii) $d(Ax,By) \leq \phi(M(x,y))$ whenever $M(x, y) > 0$.

Let S be compatible with A and T be compatible with B. If the mappings in one of the compatible pairs (A,S) or (B,T) be reciprocally continuous then A, B, S and T have a unique common fixed point.

Proof. Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by

$$y_{2n} = Ax_{2n} = Tx_{2n+1}, \; y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}.$$

This can be done by virtue of (i), Then, using (ii) we obtain

(i) $d(y_{2n}, y_{2n+1}) \leq \phi(d(y_{2n-1}, y_{2n})) < d(y_{2n-1}, y_{2n})$ and

(ii) $d(y_{2n-1}, y_{2n}) \leq \phi(d(y_{2n-2}, y_{2n-1})) < d(y_{2n-2}, y_{2n-1})$.

We thus see that $\{d(y_n, y_{n+1})\}$ is a strictly decreasing sequence of positive numbers and hence tends to a limit $r \geq 0$. Suppose $r > 0$. Then relation (i) on making $n \to \infty$ and in view of upper semicontinuity of ϕ yields $r \leq \phi(r) < r$, a contradiction.

Hence $r = lim_{n\to\infty}\, d(y_n,y_{n+1}) = 0$. We thus show that $\{y_n\}$ is a Cauchy sequence. Suppose it is not. Then there exist an $\in > 0$ and a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $d(y_{n_i}, y_{n_{i+1}}) > 2\in$. Since $lim_{n\to\infty}\, d(y_n, y_{n+1}) = 0$, there exist integers m_i satisfying $n_i < m_i < n_i+1$ such that $d(y_{n_i}, y_{m_i}) \geq \in$. If not, then

$$d(y_{n_i}, y_{n_{i+1}}) \leq d(y_{n_i}, y_{n_{i+1}-1}) + d(y_{n_{i+1}-1}, y_{n_{i+1}})$$
$$< \in + d(y_{n_{i+1}-1}, y_{n_{i+1}}) < 2\in,$$

a contradiction. If m_i be the smallest integer such that $d(y_{n_i},y_{m_i}) \geq \in$, then

$$\in \leq d(y_{n_i},y_{m_i}) \leq d(y_{n_i},y_{m_i-2}) + d(y_{m_i-2}, y_{m_i-1}) + d(y_{m_i-1}, y_{m_i})$$
$$< \in + d(y_{m_{i-2}},y_{m_{i-1}}) + d(y_{m_{i-1}}, y_{m_i}).$$

That is there exist integers m_i satisfying $n_i < m_i < n_{i+1}$ such that

$d(y_{n_i}, y_{m_i}) \geq \in$ and

(3) $\quad lim_{\ n_i \to \infty} d\ (y_{n_i}, y_{m_i}) = \in$.

Without loss of generality we can assume that n_i is odd and m_i even. Now, by virtue of (1), we have

$$d\ (y_{\ n_{i+1}}, y_{m_{i+1}}) \leq \phi\ (max\ \{d\ (y_{n_i}, y_{m_i})\}\ .$$

Now on letting $n_i \to \infty$ and in view of (3) and upper semicontinuity of ϕ, the above relation yields $\in \leq \phi\ (\in)\ < \in$, a contradiction. Hence $\{y_n\}$ is a Cauchy sequence. Since X is Complete, there exists a point z in X such that $y_n \to z$. Also

$y_{2n} = Ax_{2n} = Tx_{2n+1} \to z$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \to z$.

Suppose A and S are compatible and reciprocally continuous mappings. Then, reciprocal continuity of A and S implies that $ASx_n \to Az$ and $SAx_n \to Sz$. Compatibility of A and S yields $lim_n\ d(AS_n, SAx_n) = 0$, that is, $d\ (Az,Sz) = 0$. Hence $Az = Sz$. Since $AX \subset TX$, there exists a point w in X such that $Az = Tw$. Using (ii) we get

$$d\ (Az, Bw) \leq \phi\ (max\ \{d(Sz,Tw), d(Az,Sz), d\ (Bw, Tw)\}$$
$$\leq \phi\ (d\ (Bw, Az) < d\ (Bw, Az),$$

a contradiction. Hence $Az = Bw$ and $Az = Bw = Sz = Tw$. Compatibility of A and S implies that $ASz = SAz$ and, therfore $AAz = ASz = SAz = SSz$. Similarly, pointwise *R-weak* commutativity of B and T implies that there exists $R > 0$ such that $d\ (BTw, TBw) \leq R\ d\ (Bw, Tw) = 0$, that is, $BTw = TBw$ and $BBw = BTw = TBw = TTw$. Using again (ii) we get

$$d\ (Az, AAZ) = d\ (Bw, AAz) = d(AAz, Bw)$$
$$\leq \phi\ (max\ \{d\ (SAz, Tw), d\ (AAz, SAz), d(Bw, Tw\})$$
$$= \phi\ (d(AAz,Az)) < d(AAz, Az),$$

a contradiction. Hence $Az = AAz = ASz$. Thus Az is a common fixed point of A and S. Similarly by using (ii) we obtain Bw is a common fixed point of B and T. Hence $Az = Bw$ is a unique common fixed point of A,B,S and T. The proof is similar when B and T are assumed compatible and reciprocally continuous. This completes the proof of the theorem. To consider an example for this theorem, we may define (X, d), A, B,S and T as in the above example and let $\phi\ (t) = ht$, $0 \leq h < 1$, then we obtain the Theorem 1 as a special case of the present theorem.

Discussion

Theorem 3.3 of Jachymaski [2] assumes one of the mappings to be continuous while Theorem 5.1 of the same paper assumes S and T to be continuous. The main theorem of Rhoades *et. al.* [8] and Jungck *et al.* [4] requires S and T to be continuous. Likewise, the theorems of Fisher [1] and Pant [6], [7] assume on of the mappings to be continuous. Here it is clear from the above example 1.1 that the present theorems do not require one of the mappings in a compatible pair or any other mapping to be continuous.

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