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HYDROMAGNETIC THERMAL INSTABILITY OF RIVLIN-ERICKSEN COMPRESSIBLE FLUID IN THE PRESENCE OF HALL-CURRENT

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ABSTRACT

The thermal instability of elastico-viscous Rivlin-Ericksen compressible fluid is considered in the presence of uniform vertical magnetic field to include the Hall-current. It is found that the stability criterion is independent of the effects of viscosity and viscoelasticity and is dependent on the magnitude of the magnetic field and Hall-current. The magnetic field is found to stabilize a certain wavenumber range of the unstable configuration. The system is found to be stable for $(C_p\beta)/g<1$; Cp, β and g being specific heat at constant pressure, uniform adverse temperature gradient and acceleration due to gravity. For $(C_p\beta)/g>1$ the compressibility, magnetic field and Hall-current are found to having stabilizing effects. The uniform vertical magnetic field and Hall current introduce oscillatory modes in the system which were non-existent in their absence for $(C_p\beta)/g>1$. The sufficient conditions for the non-existence of overstability, in the presence of magnetic field and Hall-current on the problem are also derived.

1. Introduction. The theory of thermal instability of a fluid layer heated from below under varying hydromagnetic assumptions was summarized by Chandrasekhar [1]. A good account of hydrodynamic stability problems has also been given by Drazin and Reid [2] and Joseph [6]. Gupta [5] studied the thermal instability of fluid in the presence of Hall currents. The Boussinesq approximation was used in all the above studies. The approximation is well justified in the case of incompressible fluids.

When the fluids are compressible, the equations governing the system become quite complicated. To simlify them, Bousssinesq tried to justify the approximation for compressible fluids for density variations arising principally from thermal effects. Spiegel and Veronis [10] simplified the set of equations governing the flow of compressible fluids under the following assumptions :

a. The vertical dimension of the fluid is much less than any scale height, as defined by Spiegel and Veronis [10] ,

b. The motion-induced perturbations in density and pressure do not exceed, in order of magnitude, their total static variations.

Under the above approximations, Spiegel and Veronis [10] showed that the equations governing convection in a compressible fluid are formally equivalent to those for an incompressible fluid if the static temperature gradient is replaced by its excess over the adiabatic one and C_v is replaced by C_p ; where C_v and C_p are the specific heats at constant volume and constant pressure respectively.

Sharma [8] has studied the stability of a layer of electrically conducting Oldroyd fluid (i.e. fluid described by Oldroyd [7] constitutive relation) heated from below in the presence of magnetic field and has found that the magnetic field has a stablizing effect.

In another study the stability of the plane interface separating two viscoelastic (Oldroyd) superposed fluids of uniform densities. Fredricksen [3] has given a good review of non-Newtonian fluids whereas Joseph [6] has also considered the stability of viscoelastic fluids. There are many non-Newtonian fluids that cannot be characterized by Oldroyd's [7] constitutive relations. The Rivlin-Ericksen elastico-viscous fluid is one such fluid. Garg, Srivastava and Singh [4] have studied the drag on sphere oscillating in conducting dusty Rivlin-Ericksen elastico-viscous liquid. Sharma and Kumar [9] have also studied hydromagnetic stability of two Rivlin-Ericksen elastico-viscous superposed conducting fluids. It is this class of elastico-viscous fluids we are interested particularly to study the effect of Hall-current on the thermal instability of Rivlin-Ericksen compressible fluid pervaded by a uniform horizontal magnetic field in addition to a constant gravity field; which is important in ground water hydrology, chemical engineering, modern technology and industries. This aspect forms the subject of the present paper.

2. Formulation of the Problem and perturbation equations.

Let T_{ij}, τ_{ij}, e_{ij}, δ_{ij}, p, q_i, x_i, μ and μ' denote respectively the stress tensor, shear stress tensor, rate of strain tensor, kronecker delta, scalar pressure, velocity position vector, viscosity and viscoelasticity. Then the Rivlin-Ericksen elastico-viscous fluid is described by the constitutive relations

$$\left.\begin{aligned} T_{ij} &= -p\delta_{ij} + \tau_{ij} \\ \tau_{ij} &= 2\left[\mu + \mu' \frac{d}{dt}\right] e_{ij} \\ e_{ij} &= \frac{1}{2}\left[\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right] \end{aligned}\right\}, \tag{1}$$

Consider an infinite horizontal compressible elastico-viscous and finitely (electrically) conducting Rivilin-Ericksen fluid layer of depth d in which a uniform temperature gradient, $\beta = |dT/dz|$ is maintained. Consider certesian coordinates (x,y,z) with origin on the lower boundary $z = 0$ and the z-*axis* perpendicular to it along the vertical. A gravitational field $\vec{g} = (0, 0, -g)$ and uniform vertical magnetic field $\vec{H} = (0, 0, H)$ pervade the system.

Let $\rho, p, \rho_m, T, \alpha, \kappa', \kappa = \kappa'/(\rho_m C_p), g, \upsilon = \mu/\rho_m, \upsilon' = \mu'/\rho_m, \bar{q}\,(u, v, w), \eta, N_e$ and e denote respectively the density, scalar pressure, constant spatial average of density, temperature, thermal coefficient of expansion, thermal conductivity, thermal diffusivity, gravitational acceleration, kinematic viscosity, kinematic viscoelasticity, fluid velocity, resistivity, electron number density and electron charge.

Then the equations expressing the conservation of momentum, mass, heat and the equation of state are

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}\,.\nabla)\vec{q} = -\frac{1}{\rho_m}\nabla p + \vec{g}\frac{\rho}{\rho_m} + \frac{1}{4\pi\rho_m}(\nabla \times \vec{H}) \times \vec{H} + \left[\upsilon + \upsilon'\frac{\partial}{\partial t}\right]\nabla^2 \vec{q}, \tag{2}$$

$$\frac{\partial \rho}{\partial t} + (\vec{q}\,.\nabla)\rho + \rho\,(\nabla.\,\vec{q}\,) = 0, \tag{3}$$

$$\frac{\partial T}{\partial t} + (\vec{q}\,.\nabla)\,T = \kappa\,\nabla^2 T, \tag{4}$$

$$\rho = \rho_m\,[1 - \alpha\,(T - T_m)], \tag{5}$$

where T_m is the temperature of which $\rho = \rho_m$. The magnetic permeability has been taken to be unity.

The Maxwell's equations yield

$$\frac{\partial \vec{H}}{\partial t} = \nabla \times (\vec{q} . \vec{H}) + \eta \nabla^2 \vec{H} - \frac{1}{4\pi N_e e} \nabla \times [(\nabla \times \vec{H}) \times \vec{H}] \tag{6}$$

$$\nabla . \vec{H} = 0. \tag{7}$$

Initially, $\vec{q} = (0,0,0)$, $\vec{H} = (0,0, H)$, $T = T(z)$, $p = p(z)$, $\rho = \rho(z)$, where, following Spiegel and Veronis [10], we have

$$\left. \begin{aligned} & T(z) = -\beta z + T_0,\ p(z) = p_m - g\int_0^z (\rho_m + \rho_0) dz,\ K_m = \left[\frac{1}{\rho}\frac{\partial \rho}{\partial p}\right]_m \\ & \rho(z) = \rho_m\,[1 - \alpha_m (T - T_m) + K_m (p - p_m)], \alpha_m = -\left[\frac{1}{\rho}\frac{\partial \rho}{\partial p}\right]_m (=\alpha, say), \end{aligned} \right\} \tag{8}$$

Spiegel and Veronis (10) expressed any state variable, say X, in the form $X = X_m + X_0 (z) + X' (x,y,z,t)$,
where X_m is the constant spatial distribution of X, X_0 is the variation in X in the absence of motion and $X' (x, y, z, t)$ is the perturbations in X due to motion of the fluid. Thus p_m and ρ_m are the constant spatial distributions of p and ρ, and ρ_0 and T_0 are the density and temperatue of the fluid at the lower boundary $z = 0$. The pressure and temperature have been shown to be related by relations of the form (9) (Spiegel and Veronis [10], which have been obtained from the basic equations by integration.

Let δp, $\delta\rho$, θ, $\vec{h} = (h_x, h_y, h_z)$ and $\vec{q} = (u,v,w)$ denote respectively the perturbations in pressure p, density ρ, temperature T, magnetic field $\vec{H}(0,0,H)$ and velocity $(0,0,0)$. The change in density $\delta\rho$, caused mainly by the perturbation θ in temperature, is given by

$$\delta\rho = -\rho_m \alpha\theta. \tag{9}$$

Then the linearized hydromagnetic perturbation equations appropriate for the problem, under the Spiegel and Veronis [10] approximations, are

$$\frac{\partial \vec{q}}{\partial t} = -\frac{1}{\rho_m}\nabla\delta p \times \frac{1}{4\pi\rho_m}(\nabla \times \vec{h}) \times \vec{H} - g\alpha\theta + \left[\upsilon + \upsilon'\frac{\partial}{\partial t}\right]\nabla^2 \vec{q}, \tag{10}$$

$$\nabla . \vec{q} = 0, \tag{11}$$

$$\frac{\partial \theta}{\partial t} = \left[\beta - \frac{g}{C_p}\right] w + \kappa\nabla^2\theta, \tag{12}$$

$$\frac{\partial \vec{h}}{\partial t} = \nabla \times (\vec{q} . \vec{H}) + \eta \nabla^2 \vec{h} - \frac{1}{4\pi N_e e} \nabla \times [(\nabla \times \vec{H}) \times \vec{H}], \tag{13}$$

$$\nabla.\vec{h} = 0, \tag{14}$$

where g/C_p is the adiabatic gradient, Under the above mentioned approximations, as shown by Spiegel and Veronis [10], the equations governing convection in a compressible fluid have been written as formally equivalent to those for an incompressible fluid, except that the static temperature gradient β is replaced by $\beta - g/C_p$.

3. Dispersion relation. Analyzing the disturbances into normal modes, we assume that the perturbation quantities are of the form

$$[w, \theta, h_z, \zeta, \xi] = [W(z), \Theta(z), K(z), Z(z), X(z)]\ exp\ (ik_x x + ik_y y + nt) \tag{15}$$

where k_x and k_y are the wave numbers along the x-and y-directions. $k = (k_x^2 + k_y^2)^{½}$ is the resultant wave number and n is, in general, a complex constant. $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ and $\xi = \frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}$ are the z-components of the vorticity and current density respectively.

Expressing the coordinates (x,y,z) in the new unit of length d and letting $a = kd$, $\sigma = nd^2/\upsilon$, $p_1 = \upsilon/\kappa$, $p_2 = \upsilon/\eta$, $\upsilon'' = \upsilon'/d^2$ and $D = d/dz$ equaions (10)-(14), with the help of (9) and (15), in non-dimensional form become

$$[\sigma - (1+\upsilon''\sigma)(D^2-a^2)](D^2-a^2)W + \frac{g\alpha d^2}{\upsilon} a^2\Theta - \frac{Hd}{4\pi\rho_m\upsilon}(D^2-a^2)DK = 0, \tag{16}$$

$$[\sigma - (1+\upsilon''\sigma)(D^2-a^2)]Z = \frac{Hd}{4\pi\rho_m\upsilon}DX, \tag{17}$$

$$(D^2-a^2-p_2\sigma)X = -\frac{Hd}{\eta}DZ - \frac{H}{4\pi N_e e\eta d}(D^2-a^2)DK, \tag{18}$$

$$(D^2-a^2-p_2\sigma)K = -\frac{Hd}{\eta}DW - \frac{H}{4\pi N_e \eta}DX, \tag{19}$$

$$(D^2-a^2-p_1\sigma)\Theta = -\frac{d^2}{\kappa}\left(\beta - \frac{g}{C_p}\right)W. \tag{20}$$

Eliminating Θ, K, X and Z between (17)-(20) we obtain

$[(D^2-a^2-p_2\sigma)^2\{\sigma - (1+\upsilon''\sigma)(D^2-a^2)\} + QD^2(D^2-a^2-p_2\sigma) + MD^2(D^2-a^2)$
$\{\sigma-(1+\upsilon''\sigma)(D^2-a^2)\}] \times [(D^2-a^2)(D^2-a^2-p_1\sigma)\{\sigma-(1+\upsilon''\sigma)(D^2-a^2)\} - Ra^2$
$\left(\frac{G-1}{G}\right)$ $]\ W + Q(D^2-a^2)(D^2-a^2-p_1\sigma) + \Big[\ [\{\sigma - (1+\upsilon''\sigma)(D^2-a^2)\}(D^2-a^2-p_2\sigma)$

$$+QD^2\}]\, D^2 \Big] W = 0, \tag{21}$$

where $Q = \dfrac{H^2 d^2}{4\pi\rho_m \upsilon\eta}$ is the Chandrasekhar number, $R = \dfrac{g\alpha\beta d^2}{\upsilon\kappa}$ is the Rayleigh number, $M = \dfrac{H}{4\pi N_e e\eta}$ is the non-dimensioanal number accounting for Hall current and $G = \dfrac{C_p\beta}{g}$. Consider the case where both the boundaries are free and the medium adjoining the fluid is non-conducting. The appropriate boundary conditions for this case are {Chandrasekhar [1]}.

$$W = D^2W = 0,\ \ \Theta = 0,\ DZ = 0,\ X = 0 \text{ at } z = 0 \text{ and } 1 \tag{22}$$

and $\vec{h}$ are continuous.

The case of two free boundaries, although rather artificial, is the most appropriate for the stellar atmospheres {Spiegel [11]}. Using the boundary conditions (22), one can show that all the even-order derivatives of W must vanish for $z = 0$ and 1, and hence the proper solution of (21) characterizing the lowest mode is

$$W = W_0 \sin \pi z, \tag{23}$$

where W_0 is a constant. Substituting (23) into (21) and letting, $a^2 = \pi^2 b^2$, $R_1 = R/\pi^4$, $Q_1 = Q/\pi^4, \bar{\upsilon} = \pi^2\upsilon$" and $i\sigma_1 = \sigma/\pi^2$ we obtain the dispersion relation

$$R = \left[\frac{G}{G-1}\right]\Big[\frac{1+b}{b}\{i\sigma_1 + (1+b)(1 + i\sigma_1\bar{\upsilon})\}(1 + b + ip_1\sigma_1) + \frac{1}{b}\{Q_1(1+b)$$
$$(1+b+ip_1\sigma_1)\ \ [\{i\sigma_1 + (1+b)(1 + i\sigma_1\bar{\upsilon})\}(1 + b + ip_2\sigma_1) + Q_1]\} \times [(1 + b + ip_2\sigma_1)^2$$
$$\{i\sigma_1 + (1+b)(1 + i\sigma_1\bar{\upsilon})\} + Q_1(1 + b + ip_2\sigma_1) + M(1+b)\{i\sigma_1 + (1+b)(1 + i\sigma_1\bar{\upsilon})\}]^{-1}\Big] \tag{24}$$

4. **Stationary convection.** When instability sets in as stationary convection, the marginal state will be characterized by $\sigma = 0$. Putting $\sigma = 0$, the dispersion relations (24) reduces to

$$R_1 = \frac{G}{(G-1)}\left[\frac{(1+b)^3}{b} + \frac{1}{b}\{Q_1(1+b)((1+b)^2 + Q_1)\} \times \{(1+b)^2 + Q_1 + M(1+b)\}^{-1}\right] \tag{25}$$

which expresses the modified Rayleigh number R_1 as a function of the dimensionless wave number b and the parameters Q_1, M and G for fixed Q_1 and M, let G (accounting for the compressibility effect)also be kept fixed. Then we find that

$$\bar{R}_c = \frac{G}{G-1} R_c , \tag{26}$$

where $\bar{R}_c$ and R_c denote respectively the critical Rayleigh numbers in the presence and absence of compressibility .The effct of compressibility is thus to postpone the onset of thermal instability. Hence compressibility has a stabilizing effect . $G > 1$ is relevant here. The cases $G < 1$ and $G = 1$ correspond to negetive and infinite values of the critical rayleigh numbers in the presenc of compressibility which are not relevant in the present study.

It follows from (25) that

$$\frac{dR_1}{dQ_1} = \frac{G}{(G-1)\, b} \left[(1+b)\{(1+b)^2+Q_1\}+Q_1(1+b)\right]\times\left[\,((1+b)^2 + Q_1 + M\,(1+b)\right]^{-2} \tag{27}$$

This shows that the magnetic field has a stabilizing effect for all wavenumber range for $G <1$ and destabilizing effect for $G >1$. in the presence of Hall current for $G > 1$, the magnetic field may have stabilizing or destabilizing effect .

Equation (25) also yields

$$\frac{dR_1}{dM} = \frac{G}{(1-G)b} Q_1 (1+b)^2\{(1+b)^2+Q_l\}\{(1+b)^2+Q_l + M(1+b)\}^{-2} . \tag{28}$$

It is evident from (28), that dR_l/dM is always positive for $G < 1$, implying thereby the stabilizing effect of Hall current and dR_l/dM is always negative for $G > 1$, implying thereby the destabilizing effect of Hall current for all wavenumbers.

We thus conclude that the presence of viscoelasticity does not effect the stability or instability of the system.

5. Stability of The System and Oscillatory Modes

Multiplying by W^*, the complex conjugate of W, integrating over the range of z , and making use of (17) - (20) , we obtain

$$[\sigma I_1 + (1+\upsilon''\sigma)I_2] + \frac{C_p \alpha\kappa a^2}{\upsilon\,(1-G)} (I_3 + p_1\sigma^* I_4) + \frac{\eta d^2}{4\pi\, \rho_m \upsilon} (I_5 + p_2\sigma I_6) +$$

$$\frac{\eta}{4\pi\, \rho_m \upsilon} (I_7 + p_2\sigma^* I_8) + d^2\,[\sigma^* I_{10} + (1+\upsilon''\sigma^*)I_9] = 0, \tag{29}$$

where

$$I_1 = \int_0^1 (|DW|^2 + a^2\,|W|^2)\, dz, \qquad I_2 = \int_0^1 (|D^2W|^2 + a^4\,|W|^2 + 2a^2\,|DW|^2) dz,$$

$$I_3 = \int_0^1 (|D\Theta|^2 + a^2 |\Theta|^2)\, dz, \quad I_4 = \int_0^1 |\Theta|^2 dz \quad I_5 = \int_0^1 (|DX|^2 + a^2 |X|^2)\, dz,$$

$$I_6 = \int_0^1 |X|^2 \; dz, \qquad I_7 = \int_0^1 (|D^2K|^2 + 2a^2 |DK|^2 + a^4 |K|^2) dz,$$

$$I_8 = \int_0^1 (|DK|^2 + a^2 |K|^2)\, dz, \qquad I_9 = \int_0^1 (|DZ|^2 + a^2 |Z|^2)\, dz,$$

$$I_{10} = \int_0^1 |Z|^2 \; dz, \tag{30}$$

which are all positive definite .The real and imaginary parts of (29) give

$$\sigma_r \left[I_1 + \upsilon'' I_2 + \frac{C_p \alpha \kappa a^2}{\upsilon (1-G)} p_1 I_4 + \frac{\eta p_2}{4\pi \rho_m \upsilon} (I_8 + d^2 I_6) + d^2 (I_{10} + \upsilon'' I_9) \right]$$

$$= - \left[I_2 + \frac{C_p \alpha \kappa a^2}{\upsilon (1-G)} I_3 + \frac{\eta}{4\pi \rho_m \upsilon} (I_7 + d^2 I_5) + d^2 I_9 \right], \tag{31}$$

$$\sigma_i \left[I_1 + \upsilon'' I_2 - \frac{C_p \alpha \kappa a^2}{\upsilon (1-G)} p_1 I_4 + \frac{\eta p_2}{4\pi \rho_m \upsilon} (d^2 I_6 - I_8) + d^2 (-I_{10} + \upsilon'' I_9) \right] = 0 \tag{32}$$

It follows from (31) that σ_r is negative if $G < 1$. The system is therefore stable for $G < 1$. It is evident from (32) that σ_i may be zero and non zero . Thus the modes may be non -oscillatory or oscillatory .The oscillatory modes are introduced owing to the presence of a magnetic field (and hence Hall current).In the absence of the magnetic field , the oscillatory modesare not allowed for $G > 1$, but the presence of a magnetic field and Hall currents introduce oscillatory modes in the system.

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