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A GENERAL FIXED POINT THEOREM FOR SOME COMPLETE METRIC SPACES

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ABSTRACT

We establish a general fixed point theorem which includes results of Khan, Swaleh and Sessa [1] and H.K. Pathak and Rekha Sharma [2] as special cases.

Let T be a selfmap of a metric space, and define $x_{n+1} = Tx_n$, $n \geq 0$, $O(x_0) := \{x_0, x_1, \dots, x_n, \dots\}$. Throughout this paper $\psi: R^+ \to R^+$ is an increasing continuous function with $\psi(t) = 0$ if and only if $t = 0$.

Theorem Let (X, d) be a metric space, T a selfmap of X. Let $a : R^+/\{0\} \to [0,1)$ be a decreasing map with $a(t) > 0$ for each $t > 0$. Suppose that there exists an $x_0 \varepsilon X$ such that $\overline{O(x_0)}$ is complete. If

(1) $\psi[d(Tx, Ty)] \leq a(d(x,y))\, \psi d(x, y))$

for each $x, y \in O(x_0)$, $x \neq y$, then T has a fixed point in $\overline{O(x_0)}$.

Proof. We may assume that $x_n \neq x_{n+1}$ for each n since, otherwise, T has a fixed point. Define $\tau_n = d(x_n, x_{n+1})$. Then, from (1),

(2) $\psi(\tau_{n+1}) \leq a(\tau_n)\psi(\tau_n) < \psi(\tau_n)$.

Since ψ is increasing, (2) implies that $\tau_{n+1} \leq \tau_n$. therefore $lim\ \tau_n = \tau \geq 0$.

Suppose that $\tau > 0$. Then, since a is decreasing, $a(\tau_n) \leq a(\tau)$ and, from (2) $\psi(\tau_{n+1}) \leq a(\tau_n)\ \psi(\tau_n)$. Taking the limit as $n \to \infty$ gives $\psi(\tau) \leq a(\tau)\psi(\tau) < \psi(\tau)$, a contradiction. Therefore $\tau = 0$.

We now wish to show that $\{x_n\}$ is Cauchy. It will be sufficient to show that $\{x_{2n}\}$ is Cauchy. Suppose that $\{x_{2n}\}$ is not Cauchy. Then there exists an $\varepsilon > 0$ such that, for each even integer $2k$, there exist subsequences of even integers $\{2m(k)\}$, $\{2n(k)\}$ such that

(3) $d(x_{2m(k)}, x_{2n(k)}) \geq \varepsilon$ for $\{2m(k)\} > \{2n(k)\} \geq 2k$.

By the well-ordering principle, for each even integer $2k$, let $2m(k)$ be the smallest even integer exceeding $2n(k)$ satisfying (3) ; i.e.

(4) $d(x_{2n(k)}, x_{2m(k)-2}) < \varepsilon$ and $d(x_{2m(k)}, x_{2n(k)}) \geq \varepsilon$.

Then, for each integer $2k$,

(5) $\varepsilon \leq d(x_{2m(k)}, x_{2n(k)-2}) \leq d(x_{2n(k)}, x_{2m(k)-2}) + (\tau_{m(k)-2} + \tau_{2m(k)-1}$.

Taking the limit of (5) as $k \to \infty$, and using 94) yields

(6) $lim_k\, d(x_{2n(k)+1}, x_{2m(k)+1}) = \varepsilon$.

$\varepsilon \leq d(x_{2m(k)}, x_{2n(k)}) \leq \tau_{2n(k)} + d(x_{2n(k)+1}, x_{2m(k)+1}) + \tau_{2m(k)}$.

Therefore

$\varepsilon - \tau_{2n(k)} - \tau_{2m(k)} \leq d(x_{2n(k)+1}, x_{2m(k)+1}) \leq \tau_{2n(k)} + d(x_{2m(k)}, x_{2n(k)}) + \tau_{2m(k)}$.

which on taking the limit as $k \to \infty$, gives

(7) $lim_k\, d(x_{2n(k)+1}, x_{2m(k)+1}) = \varepsilon$.

Substituting into (1) with $x = x_{2n(k)}$, $y = x_{2m(k)}$ we have

$$\psi(d(x_{2n(k)+1}, x_{2m(k)+1})) \leq a(d(x_{2n(k)}, x_{2m(k)}))\ \psi(d(x_{2n(k)}, x_{2m(k)}))$$
$$\leq a(\varepsilon)\ \psi(d(x_{2n(k)}, x_{2m(k)})).$$

Taking the limit of the above inequality as $k \to \infty$, and using (6) and (7), we derive

$$\psi(\varepsilon) \leq a(\varepsilon)\ \psi(\varepsilon) < \psi(\varepsilon) ,$$

a contradiction. Therefor $\{x_n\}$ is Caushy. Sinc $\overline{O(x_0)}$ is complete, there exists a number $z = lim\, x_n$.

Since $\tau_n = 0$ for each n, there exists an infinite subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \neq z$ for each k. Using (1),

(8) $\psi(d(x_{2n(k)+1}, Tz)) = \psi(d(Tx_{n(k)}, Tz)) \leq a(d(x_{n(k)}, z))\ \psi(d(x_{n(k)}, z))$
$< \psi(d(x_{n(k)}, z)).$

Taking the limit of (8) as $k \to \infty$,we have

(9) $lim_k\, \psi(d(x_{2n(k)+1}, Tz)) \leq lim_k\, \psi(d(x_{2n(k)}, z)) = \psi(0) = 0.$

Using the triangular inequality,

$d(z, Tz) \leq d(z, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1}) + d(Tx_{n(k)}, Tz)).$

Since ψ is increasing,

(10) $\psi d(z, Tz) \leq \psi d(z, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1}) + d(Tx_{n(k)}, Tz)).$

Taking the limit of (10) $k \to \infty$, and using (9) gives

$$\psi(d(z, Tz)) \leq lim_k\, \psi(d(x_{2n(k)}, z)) = 0,$$

which implies that $d(z, Tz) = 0$, and hence that $z = Tz$.

Corollary 1. Let (X, d) be a complete metric space, T a selfmap of X, ψ as in the Theorem. Let a, b be two decreasing functions from $R^+/\{0\} \to [0,1)$ such that $a(t) + b(t) < 1$ for each $t>0$. Suppose that T satisfies

(11) $\psi[d(Tx, Ty)] \leq a(d(x,y)) \max \{\psi(d(x, y)), [\psi(d(x, y)).\psi(d(y, Tx))^{1/2}\}$
$+ b(d(x,y)) \min \{\psi(d(x, Tx)), \psi(d(y, Ty))\}$,

where $x,y \in X$. then T has a unique fixed point.

Proof. We first show that (11) implies (1). In (11) set $x = x_n$, $y = x_{n+1}$ to obtain

(12) $\psi(\tau_{n+1}) \leq a(\tau_n)\psi(\tau_n) + b(\tau_n)\psi (\min \{\psi(\tau_n), \psi(\tau_{n+1})\})$,

and assume that each $\tau_n = 0$. Suppose that there exists an n such that $\psi(\tau_n) < \psi(\tau_{n+1})$. Then it follows from (12) that

$$\psi(\tau_{n+1}) \leq [a(\tau_n)+ b(\tau_n)] \psi (\tau_n) < \psi(\tau_{n+1}),$$

a contradiction. Therefore $\psi (\tau_{n+1}) \leq \psi(\tau_n)$ for each n.

From (12),

$$\psi (\tau_{n+1}) \leq \frac{a(\tau_n)}{1-b(\tau_n)} \psi(\tau_n)$$

and (1) is satisfied, since $f(t) := a(t)/(1-b(t))$ is decreasing in t for $t>0$. Also, the range of f is constained in $[0,1)$.

From the Theorem, T has a fixed point. Condition (11) implies uniqueness.

Corollary 2. [1, Theorem 2] let a, b, c be three decreasing functions from $R^+/\{0\} \to [0,1)$ such that $a(t) + b(t)+ c(t) < 1$ for each $t>0$. Suppose that T satisfies

$$\psi(d(Tx, Ty)) \leq a(d(x,y))\psi(d(x, y))+ b(d(x,y))\{\psi(d(x, Tx) + \psi(d(y, Ty))\}$$
$$+ c(d(x,y))\min \{\psi(d(x, Ty)), \psi(d(y, Tx))\}$$

for each $x,y \in X$. $x \neq y$. Then T has a unique fixed point.

Proof. With x_n, τ_n as defined in Corollary 1, we have

$$\psi(\tau_{n+1}) \leq a(\tau_n)\psi(\tau_n) + b(\tau_n)\{\psi(\tau_n), \psi(\tau_{n+1})\},$$

which implies that

$$\psi (\tau_{n+1}) \leq \frac{a(\tau_n) + b(\tau_n)}{1- b(\tau_n)} \psi(\tau_n) .$$

Let $s < t$. Then $a (s) + b (s) \geq a(t) + b(t)$. Also $b(s) \geq b(t)$, which implies that $1- b (s) \leq 1- b(t)$, and hence

$$\frac{1}{1-b(s)} \geq \frac{1}{1-b(t)} .$$

Corollary 1. Let (X, d) be a complete metric space, T a selfmap of X, ψ as in the Theorem. Let a, b be two decreasing functions from $R^+/\{0\} \to [0,1)$ such that $a(t) + b(t) < 1$ for each $t>0$. Suppose that T satisfies

(11) $\psi[d(Tx, Ty)] \le a(d(x,y)) \max \{\psi(d(x, y), [\psi(d(x, y).\psi(d(y, Tx))^{1/2}\}$
$+ b(d(x,y)) \min \{\psi(d(x, Tx), \psi(d(y, Ty))\},$

where $x,y \in X$. then T has a unique fixed point.

Proof. We first show that (11) implies (1). In (11) set $x = x_n$, $y = x_{n+1}$ to obtain

(12) $\psi(\tau_{n+1}) \le a(\tau_n)\psi(\tau_n) + b(\tau_n)\psi\ (\min \{\psi(\tau_n), \psi(\tau_{n+1})\}),$

and assume that each $\tau_n = 0$. Suppose that there exists an n such that $\psi(\tau_n) < \psi(\tau_{n+1})$. Then it follows from (12) that

$$\psi(\tau_{n+1}) \le [a(\tau_n) + b(\tau_n)]\, \psi\ (\tau_n) < \psi(\tau_{n+1}),$$

a contradiction. Therefore $\psi\ (\tau_{n+1}) \le \psi(\tau_n)$ for each n.

From (12),

$$\psi\ (\tau_{n+1}) \le \frac{a(\tau_n)}{1\text{-}b(\tau_n)}\ \psi(\tau_n)$$

and (1) is satisfied, since $f(t) := a(t)/(1-b(t))$ is decreasing in t for $t>0$. Also, the range of f is constained in $[0,1)$.

From the Theorem, T has a fixed point. Condition (11) implies uniqueness.

Corollary 2. [1, Theorem 2] let a, b, c be three decreasing functions from $R^+/\{0\} \to [0,1)$ such that $a\ (t) + b\ (t) + c\ (t) < 1$ for each $t>0$. Suppose that T satisfies

$\psi(d(Tx, Ty)) \le a(d(x,y))\psi(d(x, y)) + b(d(x,y))\{\psi(d(x, Tx) + \psi(d(y, Ty))\}$
$+ c(d(x,y))\min \{\psi(d(x, Ty), \psi(d(y, Tx))\}$

for each $x,y \in X$. $x \ne y$. Then T has a unique fixed point.

Proof. With x_n, τ_n as defined in Corollary 1, we have

$$\psi(\tau_{n+1}) \le a(\tau_n)\psi(\tau_n) + b(\tau_n)\{\psi(\tau_n), \psi(\tau_{n+1})\},$$

which implies that

$$\psi\ (\tau_{n+1}) \le \frac{a(\tau_n) + b(\tau_n)}{1 - b(\tau_n)}\ \psi(\tau_n)\ .$$

Let $s < t$. Then $a\ (s) + b\ (s) \ge a(t) + b(t)$. Also $b(s) \ge b(t)$, which implies that $1 - b\ (s) \le 1 - b(t)$, and hence

$$\frac{1}{1-b(s)} \ge \frac{1}{1-b(t)}\ .$$

Therefore $(a(t) + b(t))/(1-b(t))$ is a decreasing function of t and its range is included in $[0,1)$. Theorem 1 then applies to give a fixed point. The contractive condition implies uniqueness.

Corollary 3. [2, Theorem 2] let a, b, be two decreasing functions from $R^+/\{0\} \to [0,1)$ such that $a(t) + b(t) < 1/2$ for each $t>0$. Suppose that T satisfies

$$\psi[d(Tx\,,\,Ty)] \le a(d(x,y))\{\psi(d(x\,,y)) + c[\psi(d(x,y))\psi(d(y\,,Tx)]^{1/2}\} + b(d(x,y))\{\psi(d(x\,,Tx) + \psi(d(y\,,\,Ty))\}$$

for each $x,y \in X$. where $c \in [0,1]$ such that $a(t)\,(1+c) < 1$. Then T has a unique fixed point.

Proof. With x_n, τ_n as in Corollary 1, we have

$$\psi(\tau_{n+1}) \le a(\tau_n)\psi(\tau_n) + b(\tau_n)\{\psi(\tau_n) + \psi(\tau_{n+1})\},$$

which implies that

$$\psi\,(\tau_{n+1}) \le \frac{a(\tau_n) + b(\tau_n)}{1-b(\tau_n)}\,\psi(\tau_n)\,.$$

As in the proof of Corollary 2, $(a(t) + b(t))/(1-b(t))$ is a decreasing function of t with range included in $[0,1)$. Therefore, from Theorem 1, T has a fixed point. The contractive condition implies uniqueness.

Note that the conditions on c are not needed.

With ψ the identity mapping, $a \equiv k$ one obtains the Banch contraction principle.

With ψ the identity mapping, one obtains the result of Rakotch[3].

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