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HEAT CONDUCTION AND A GENERAL CLASS OF POLYNOMIALS

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ABSTRACT

The object of this paper is to use a general class polynomials in finding the solution of the boundary value problems concerning heat conduction. Some special examples are also considered.

1. Introduction. The Hermite polynomials have been utilised by Kampé de Fériet [4] in solving a heat conduction equation. He has obtained four theorems wich are of the nature of existence theorems. Problems of heat conduction and heat production are also attempted in two well quoted papers by Bhonsle [1,2] .

Here we solve the problem of heat conduction in the case of a non–homogeneous bar and a circular clyinder.

Consider the partial differential eqation

$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial u^2} - kVu^2 \qquad \text{... (1.1)}$$

Under some suitable initial and boundary conditions, eq (1.1) can be considered as a heat conduction equation

$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial u^2} - h\,(V - V_0) \qquad \text{... (1.2)}$$

provided $V_0 = 0,\ h = ku^2$.

When $V(u, t) \to 0$, both for the large values of t and $|u|$. we may assume the solution of (1.1) to be

$$V(u\,,t) = \sum_{y=0}^{\infty} D_y\, e^{-(1+2y)kt}\, H_y(u) \qquad \text{... (1.3)}$$

where $H_y(u)$ denotes the well known Hermite polynomial.

When $t = 0$, let $V(u\,,0) = f(u)$.

$$f(u) = u^{\alpha}\, S_n^m(\delta\, u^{2\rho}) \qquad \text{...(1.4)}$$

where $S_n^m(x) = \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}}{j!} A_{n,j}\, x^j$ is a general class of polynomials [7]

Then $f(u) = \sum_{y=0}^{\infty} D_y H_y(u) = u^{\alpha} S_n^m(\delta u^{2\rho})$ ... (1.5)

Now multiplying both sides by $e^{-u^2} H_\alpha(u)$ and integrating with respect to u from $-\infty$ to ∞ and making use of the orthogonality property for Hermite polynomials we obtain

$$\sum_{y=0}^{\infty} D_y \int_{-\infty}^{\infty} e^{-u^2} H_\alpha(u) H_y(u)\, du = \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}}{j!} A_{n,j}\, \delta^j \int_{-\infty}^{\infty} e^{-u^2} u^{\alpha+2\rho j} H_\alpha(u)\, du. \quad (1.6)$$

Evaluating the integral on the right hand Side, We obtain

$$D_\alpha = \frac{1}{2^\alpha a! \sqrt{\pi}} \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}}{j!} \frac{\Gamma(a+2\rho j+1)}{\Gamma(\rho j+1)} (\delta/2^{2\rho})^j A_{n,j} \quad \text{... (1.7)}$$

We thus have

$$V(u,t) = \frac{1}{\sqrt{\pi}} \sum_{y=0}^{\infty} \sum_{j=0}^{[n/m]} \frac{e^{-(1+2y)kt}}{2^y y!} H_y(u) \frac{(-n)_{mj}}{j!} A_{n,j} \frac{\Gamma(y+2\rho j+1)}{\Gamma(\rho j+1)} \left(\frac{\delta}{2^{2\rho}}\right)^j \quad \text{...(1.8)}$$

Particular cases

(i) If we take $m = 1$, and

$A_{n,j} = \binom{n+\alpha}{n} \frac{(\alpha+\beta+n+1)}{(\alpha+1)_j}$ then $S_n^1(x) \to P_n^{(\alpha,\beta)}(1-2x)$ and

$S_n^m(x)$ reduces to the Jacobi polynomials. Eq. (1.4) now reduces to

$f(u) = u^\alpha P_n^{(\alpha,\beta)}(1-2\delta u^{2\rho})$

and

$$V(u,t) = \binom{n+\alpha}{n} \frac{1}{\sqrt{\pi}} \sum_{y=0}^{\infty} \sum_{j=0}^{n} \frac{e^{-(1+2y)kt}}{2^y y!} H_y(u) (-n)_j \frac{\Gamma(y+2\rho j+1)}{j!\,\Gamma(\rho j+1)} \frac{(\alpha+\beta+n+1)_j}{(\alpha+1)_j} \left(\frac{\delta}{2^{2\rho}}\right)^j \quad \text{...(1.9)}$$

(ii) Similarly if we take $m=1$,

$A_{n,j} = \binom{n+\alpha}{n} \frac{1}{(\alpha+1)_j}$ and $S_n^1(x) \to L_n^\alpha(x)$ and

then $S_n^m(x)$ reduces to the Laguerre polynomials

$f(u) = u^\alpha L_n^\alpha(\delta u^{2\rho})$

and

$$V(u,t) = \binom{n+\alpha}{n} \frac{1}{\sqrt{n}} \sum_{y=0}^{\infty} \sum_{j=0}^{n} \frac{e^{-(1+2y)kt}}{2^y y!} H_y(u) \frac{(-n)_j \Gamma(y+2\rho j+1)}{j!\,\Gamma(\rho j+1)} \frac{1}{(\alpha+1)_j} \left(\frac{\delta}{2^{2\rho}}\right)^j \quad \text{...(1.10)}$$

(iii) Another interesting paricular case of the polynomials $S_n^m(x)$ is the generalized Bessel polynomials, if we take $m=1$

$$A_{n,j} = (\alpha+n-1)_j \quad , \quad S_n^1(x) \to Y_n(-\beta x, \alpha, \beta)$$

then $S_n^m(x) \to Y_n(-\beta x, \alpha, \beta) = \sum_{j=0}^{n} \binom{n}{j} \binom{n+\alpha+j-2}{j} j!\left(\frac{x}{\beta}\right)^j$

$$= {}_2F_0\left[\begin{matrix}-n,\ \alpha+n-1; \\ \text{—};\end{matrix}\ -x/\beta\right]$$

$$f(u) = u^\alpha\, Y_n(-\beta\delta u^{2\rho}, \alpha, \beta)$$

$$V(u,t) = \frac{1}{\sqrt{\pi}} \sum_{y=0}^{\infty} \sum_{j=0}^{n} \frac{e^{-(1+2y)kt}}{2^y y!} H_y(u) \frac{(-n)_j \Gamma(y+2\rho j+1)}{j!\,\Gamma(\rho j+1)} (\alpha+n-1)_j \left(\frac{\delta}{2^{2\rho}}\right)^j \quad \ldots(1.11)$$

(iv) Brafman polynomials

$$B_n^m[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;x] = {}_{m+1}F_q\left[\begin{matrix}\Delta(\mathrm{m};-\mathrm{n}),\alpha_1,\ldots,\alpha_p; \\ \beta_1,\ldots,\beta_p;\end{matrix}\ x\right]$$

for which we take in (1.4).

$$A_{n,j} = \frac{(\alpha_1)_j \ldots (\alpha_p)_j}{(\beta_1)_j \ldots (\beta_q)_j}$$

$$S_n^m(x) \to B_n^m[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;xm^m]$$

where $\Delta(m;\lambda)$ abbreviates the array of m parameters

$$f(u) = u^\alpha B_n^m[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;(\delta u^{2\rho} m^m)]$$

Therefore,

$$V(u,t) = \frac{1}{\sqrt{\pi}} \sum_{y=0}^{\infty} \sum_{j=0}^{[n/m]} \frac{e^{-(1+2y)kt}}{2^y y!} H_y(u) \frac{(-n)_{mj}}{j!} \frac{\Gamma(y+2\rho j+1)}{\Gamma(\rho j+1)} \frac{(\alpha_1)_j \ldots (\alpha_p)_j}{(\beta_1)_j \ldots (\beta_p)_j} \left(\frac{\delta}{2^{2\rho}}\right)^j$$

2. Heat conduction in a finite bar. When the lateral surface of the bar is insualted, the heat eqation has the from [3]

$$\frac{\partial V}{\partial t} = k \frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial V}{\partial x}\right] \quad \ldots (2.1)$$

where k is a constant, provided that the thermal cofficient is constant. The boundary conditions of the problem are that the both ends of the bar at $x = \pm 1$ are insulated so that the conductivity vanishes there, and the initial condition is

$$V(x,0) = f(x)\,, \qquad -1 < x < 1 \quad \ldots (2.2)$$

Further we assume,

$$V(x,0) = f(x) = (1-x)^\eta S_N^M[(1-x)^\lambda] \quad \ldots (2.3)$$

where $\lambda \geq 0$.

We may now write the solution of (2.1) in the form

$$V(x,t) = \sum_{n=0}^{\infty} R_n \, e^{-kn(n+1)t} \, P_n(x) \qquad \text{... (2.4)}$$

when $t = 0$,

$$f(x) = (1-x))^{\eta} \, S_N^M [(1-x)^{\lambda}] = \sum_{n=0}^{\infty} R_n P_n(x) \qquad \text{... (2.5)}$$

Evaluating R_n with the aid of the orthogonality property of the Legendre polynomicals and evaluating the integral and putting the value in (2.5) we get the result

$$V(x,t) = 2^{\eta} \sum_{n=0}^{\infty} \sum_{j=0}^{[N/M]} \frac{(-N)_{Mj}}{j\,!} A_{n,j} \frac{[\Gamma(\eta+\lambda j)]^2}{n!\Gamma-(\eta+\lambda j)} P_n(x) \frac{\Gamma(n-\eta+\lambda j)}{\Gamma(\lambda j+\eta+n+2)} 2^{\lambda j} (2n+1)$$

$$e^{-kn(n+1)t} \qquad \text{...(2.6)}$$

Particular cases .

(1) If $M = 1$ and $A_{N,j} = \binom{N+\alpha}{N} \frac{(\alpha+\beta+N+1)_j}{(\alpha+1)_j}$.

then $S_N^m(x)$ reduces to the Jacobi Polynomials $S_N^1(x) \rightarrow P_N^{(\alpha+\beta)}(1-2x)$ and therefore now by eq. (2.5) and (2.6) for

$$f(x) = (1-x)^{\eta} \, P_n^{(\alpha,\beta)} [1-2(1-x)^{\lambda}],$$

$$V(x,t) = 2^{\eta} \binom{N+\alpha}{N} [\Gamma(n+\lambda)]^2 \sum_{n=0}^{\infty} \sum_{j=0}^{N} \frac{(-N)_j \qquad (2n+1)\Gamma(n-\eta-\lambda j)}{j\,!\,n\,!\ \Gamma-(\eta+\lambda j)\,\Gamma(\lambda j+\eta+n+2)}$$

$$\frac{(\alpha+\beta+N+1)_j}{(\alpha+1)_j} 2^{\lambda j} e^{-kn(n+1)t} P_n(x) \qquad \text{...(2.7)}$$

(ii) If $M=1$, and $A_{N,j} = (\alpha+N-1)_j$ then $S_N^1 \rightarrow Y_n(-\beta x, \alpha, \beta)$.
Therfore $S_n^m(x)$ now reduces to the Bessel Polynomials and the relevent equation reduce to

$$f(x) = (1-x)^{\eta} \, Y_N [-\beta(1-x)^{\lambda}, \alpha, \beta)$$

then

$$V(x,t) = 2^{\eta} \sum_{n=0}^{\infty} \sum_{j=0}^{N} \frac{(-N)_j}{j\,!} \frac{(\alpha+N-1)_j \, 2^{\lambda j} (2n+1)\Gamma(n-\eta-\lambda j)}{n\,!\ \Gamma-(n+\lambda j)\,\Gamma(\lambda j+\eta+n+2)} e^{-kn(n+1)t} P_n(x) \qquad \text{... (2.8)}$$

3. Production of heat in a cylinder.

The fundamental differential eqation is of the form Sneddon [5]

$$\frac{\partial V}{\partial t} = \frac{k}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r}\right) + \theta(r, t) \qquad \text{... (3.1)}$$

Assuming the cylinder to be infinitely long so that the variation with respect to z may be neglected and the rate of heat generation is independent of temperature. In addition, suppose the surface $r = a$ is maintained at zero temperature and the initial distribution of temperature is also zero. We also assume,

$$\theta(r,t) = k/K\, f(r)\, g(t) \qquad (3.2)$$

where k is the diffusivity and K the conductivity of the material. The single variable function $f(r)$ may be represented both sources and sinks embedded in the system. Whenever the product gives the negative value it should be treated as a sink. Further, if $g(t)>0$ then inner circular cylinder encloses sources while the volume between two concentric cylinders will contain the sink. If $g(t) < 0$ then source and sink will interchange roles.

Let us we write

$$f(r) = [1- r^2/a^2]^\sigma\, S_n^m(r^2/a^2) \qquad (3.3)$$

Note that $f(r) = 0$ when $r = a$

Also $f(r) = 1$ if $\sigma = 0,\ m = n = 0.$

We shall characterise the heat source by the behaviour of the function $g(t)$.

The finite Hankel transform of $f(r)$ [6] is given by

$$J[f(r)] = \int_0^a r\, f(r)\, J_0(r\xi_i)\, dr = \overline{f}_j(\xi_i) \qquad ...(3.4)$$

Therefore, we have

$$J\left[\left(1-\frac{r^2}{a^2}\right)^\sigma S_n^m\left(\frac{r^2}{a^2}\right)\right] = \frac{a^2}{2}\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}}{j!}A_{n,j}\,\frac{\Gamma(\sigma+1)\,\Gamma(j+1)}{\Gamma(j+\sigma+2)}\ {}_1F_2\left[\begin{matrix} j+1\,; \\ 1,\ j+\sigma+2; \end{matrix}\ \frac{-a^2\xi_i^2}{4}\right] \qquad ...(3.5)$$

Here, ξ_i is the root of the transcendental eqation

$$J_0(a\xi_i) = 0 \qquad ...\ (3.6)$$

Using the inversion formula [6] we thus obtain

$$\left(1-\frac{r^2}{a^2}\right)^\sigma S_n^m\left(\frac{r^2}{a^2}\right) = \sum_i\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}}{j!}A_{n,j}\,\frac{\Gamma(\sigma+1)\Gamma(j+1)}{\Gamma(j+\sigma+2)}\,\frac{J_0(r\xi_i)}{[J_1(a\xi)]^2}\ {}_1F_2\left[\begin{matrix} j+1\,; \\ 1,\ j+\sigma+2\,; \end{matrix}\ \frac{-a^2\xi_i^2}{4}\right] \qquad ...(3.7)$$

where the i^{th} sum is taken over all the positive roots of the eqation(3.6).

Solution of the problems. We apply finite Hankel transform to obtain the solution of (3.1). Using (3.3) and (3.7) the solution is obtained as [6]

$$V(r,t) = \frac{k}{K}\Gamma(\sigma+1) \sum_i \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}}{j!} A_{n,j} \frac{\Gamma(j+1)}{\Gamma(j+\sigma+2)} \frac{J_0(r\xi_i)}{[J_1(a\xi)]^2}$$

$$ {}_1F_2 \left[\begin{matrix} j+1\,; & \\ 1, j+\sigma+2\,; & \end{matrix} \frac{-a^2\xi_i^2}{4} \right] h(\xi_i, t) \qquad \ldots(3.9)$$

where i^{th} sum is taken over all the positive roots of the equation $j_0(a\xi_i)=0$ and

$$h(\xi_i, t) = \int_0^t g(\tau)\, e^{-k\xi^2_i (t-\tau)} d\tau$$

Using the same sources as used by Bhonsle [1]. We construct the following examples :

Example 1. Heat source acting for a finite interval of time

$$g(t) = \begin{cases} g_0 & 0< t < b \\ 0 & t > b \end{cases}$$

$$L\,[g(t)] = \begin{cases} g_0/p & , 0< t < b \\ g_0/p\,(1-e^{bp}) & , t > b \end{cases}$$

$$L\,[h(\xi_i, t)] = \begin{cases} \dfrac{g_0}{p(p+k\xi^2_i)} & , 0 < t< b \\ \dfrac{g_0(1-e^{bp})}{p(p+k\xi^2_i)} & , t> b\,. \end{cases}$$

Therefore

$$h(\xi_i, t) = \begin{cases} \dfrac{g_0}{k\xi^2_i} (1-e^{-kt\xi^2_i}) & , 0 \le t \le b \\ \dfrac{g_0}{k\xi^2_i} [1-e^{-kt\xi^2_i} + e^{-kt\xi^2_i(t-b)}] & , t\ge b \end{cases}$$

Substituting in (3.9), we get

$$V(r,t) = \frac{g_0}{k}\Gamma(\sigma+1) \sum_i \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}}{j!} A_{n,j} \frac{\Gamma(j+1)}{\Gamma(j+\sigma+2)} \frac{J_0(r\xi_i)}{[J_1(a\xi_i)]^2} \frac{(1-e^{-kt\xi_i})}{\xi^2_i}$$

$$ {}_1F_2 \left[\begin{matrix} j+1; & \\ 1, j+\sigma+2\,; & \end{matrix} \frac{-a^2\xi_i^2}{4} \right], 0 \le t \le b$$

$$= g_0\,\Gamma(\sigma+1) \sum_i \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}}{j!} A_{n,j} \frac{\Gamma(j+1)}{\Gamma(j+\sigma+2)} \frac{J_0(r\xi_i)}{[J_1(a\xi_i)]^2} [1-e^{-kt\xi^2_i} + e^{-k\xi^2_i(t-b)}]$$

$$ {}_1F_2 \left[\begin{matrix} j+1\,; & \\ 1, j+\sigma+2\,; & \end{matrix} \frac{-a^2\xi_i^2}{4} \right], t \ge b.$$

Obviously, $V(r,0) = 0$.

Example 2. Heat source of exponential character.

$$g(t) = g_0\, t^{\upsilon-1} e^{-\alpha t} \qquad \upsilon>0,\ \alpha>0$$

$$L[g(t)] = g_0\, \Gamma(\upsilon)/(p+\alpha)^{\upsilon}$$

$$L\,[h\,(\xi_i,\, t)] = g_0\, \Gamma(\upsilon) \sum_{r=0}^{\infty} \frac{(\alpha-k\xi_i^2)^2}{(p+\alpha)^{\upsilon+r+1}}.$$

Therefore

$$h\,(\xi_i,\, t)] = \frac{g_0\, t^{\upsilon} e^{\alpha t}}{\upsilon}\ {}_1F_1\left[\begin{matrix}1\,; \\ \upsilon+1;\end{matrix}\ (\alpha-k\xi_i^2)\, t\right],$$

$$V(r,t) = \frac{g_0}{k\upsilon}\Gamma(\sigma+1) \sum_i \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}}{j\,!} A_{n,j} \frac{\Gamma(j+1)}{\Gamma(j+\sigma+2)}\, t^{\upsilon} e^{\alpha t} \frac{J_0(r\,\xi_i)}{[J_1(a\xi_i)]^2}$$

$${}_1F_1\left[\begin{matrix}1; \\ \upsilon+1;\end{matrix}\ (\alpha-k\xi_i^2)\, t\right] {}_1F_2\left[\begin{matrix}j+1\,; \\ j+\sigma+2\,;\end{matrix}\ \frac{-a^2\xi_i^2}{4}\right]$$

Obviously, $V(r,0) = 0$

REFERENCES

[1] B.R. Bhonsle, Heat conduction and Hermite polynomials *Proc. Nat. Acad. Sci. India Sect. A* **36** (1966), 359–360.

[2] B. R. Bhonsle, Jcobi polynomials and heat conduction in a cylinder, *Math Japon.* **11** (1966), 83–90.

[3] R.V. Churchil, *Fourier Servies and Boundary Value Problems*, Mc Graw- Hill, New York, 1942.

[4] J. Kampé De Fériet, Heat Conduction and Hermite polynomials, *Bull. Calcutta. Math. Soc.* **The Golden Jubile Commenortion Volume** (1958-59), 193-204.

[5] I.N. Senddon, *Special Funtions of Mathematical Physics and Chemistry. 2nd Edition*, Liver and Boyd, London, 1964.

[6] I.N. Sneddon, *Fourier Transforms,* Mc. Graw - Hill, New York, 1957.

[7] H.M. Srivastava, The Weyl fractional integral of a general class of polynomials, *Boll. Un. Mat. Ital.* **B 2** (1983), 219-222.