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COMMON FIXED POINTS OF MAPPINGS

By

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ABSTRACT

A common fixed point theorem is obtained which gives proper generalizations of recent results due to Rhoades, Park and Moon, Jachymski, Pant, Joshi and Pande; and Pant.

1. INTRODUCTION In recent years, several authors have obtained common fixed point theorems for compatible mappings satisfying various contractive conditions. The most general among these results being those due to Jachymski [4], Pant [10], Pant, Joshi and Pande [11] and Rhoades, Park and Moon [15]. The theorems concerning sequences of mappings generally require each mapping to satisfy a compatibility condition, a condition on its range and a strong type of contractive condition . In the present paper, we obtain a common fixed point theorem involving a sequence of mappings under much weaker conditiins.

Two selfmaps A and S of a metric space (X,d) are called compatible if $lim_n d\ (ASx_n, SAx_n) = 0$ when $\{x_n\}$ is a sequence such that $lim_n Ax_n = lim_n Sx_n = t$ for some t in X. This notion was introduced by Jungck [6]. It is well known that compatibility implies commutativity at coincidence points.

Let A_1, A_2, S and T be selfmappings of a set X such that $A_1X \subset TX$ and $A_2X \subset SX$. For x_0 in X, a sequence $\{y_n\}$ defined by $y_{2n} = A_1x_{2n} = Tx_{2n+1}$ and $y_{2n+1} = A_2x_{2n+1} = Sx_{2n+2}$ is called an S, T–iteration of x_0 under A_1 and A_2.

2. Main Results. If $\{A_i\}$, i=1,2,3,..., S and T be selfmappings of a metric space (X,d), in the sequel we shall denote

$$M_{1i}\ (x,y) = max\ \{d\ (Sx,Ty),\ d(A_ix,\ Sx),\ d\ (A_i\,y,\ Ty),\ [d\ (A_1x,Ty) + d(A_i\,y,Sx)]/2\}.$$

Theorem. Let $\{A_i\}$, i=1,2,3,..., S and T be selfmappings of a complete metric space (X,d) such that

(i) $A_1X \subset TX$ and $A_2X \subset SX$

(ii) $d(A_1x, A_2y) \leq h\, M_{12}(x,y),\ 0 \leq h < 1,$

(iii) $d(A_1x, A_iy) \leq M_{1i}(x,y),$

(iv) A_1, S and A_2, T are compatible pairs of maps.

If each convergent S, T–iteration under A_1 and A_2 converges to a point in $SX \cup TX$ then all the A_i, S and T have a unique common fixed point.

Proof. Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule

$$y_{2n} = A_1 x_{2n} = Tx_{2n+1}\ ,\ y_{2n+1} = A_2X_{2n+1} = Sx_{2n+2}\ .$$

This can be done by virtue of (i). Then, by virtue of (ii), we get

$$d(y_{2n}, y_{2n+1}) \leq h\, d(y_{2n-1}, y_{2n})$$

and $d(y_{2n-1}, y_{2n}) \leq h\, d(y_{2n-2}, y_{2n-1})$.

From these inequalities we infer that $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$ Also, for any integer $p>0$, we have

$$d(y_{2n+1}, y_{2(n+p)+2}) \leq h\, d(y_{2n}, y_{2(n+p)+1}) + hd(y_{2n}, y_{2n+1}).$$

Then

$$d(y_{2n}, y_{2(n+P)+1}) \leq d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2(n+p)+2}) + d(y_{2(n+p)+2}, y_{2(n+p)+1})$$
$$< h\, d(y_{2n}, y_{2(n+p)+1}) + (2+h)\, d(y_{2n}, y_{2n+1}).$$

that is, $(1-h)\, d(y_{2n}, y_{2(n+p)+1}) < (2+h)\, d(y_{2n}, y_{2n+1})$. Similarly, $(1-h)\, d(y_{2n-1}, y_{2(n+p)}) < (2+h)\, d(y_{2n}, y_{2n-1})$. Since, $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, the above inequalities yield

$$\lim_{n\to\infty} d(y_{2n}, y_{2(n+p)+1}) = 0 = \lim_{n\to\infty} d(y_{2n-1}, y_{2(n+p)}).$$

Hence $\{y_n\}$ is a Cauchy sequence in the complete metric space X. Therefore, by our assumption, $\{y_n\}$ converges to a point in $SX \cup TX$. Let $y_n \to Su$ for some u in X. Then $y_{2n} = A_1x_{2n} = Tx_{2n+1} \to Su$ and $y_{2n+1} = A_2x_{2n+1} = Sx_{2n+2} \to Su$.

Now, using (ii) we get $d(A_1u, A_2x_{2n+1}) \leq h\, M_{12}(u, x_{2n+1})$. On letting $n\to\infty$ this inequality yields $d(A_1u, Su) \leq h\, d(A_1u, Su)$. that is $A_1u = Su$. Since the range of A_1 is contained in the range of T, there exists a point w in X such that $A_1u = Tw$. We show that $Tw = A_iw$ for each $i>1$. If $A_iw \neq Tw$ for some $i>1$, using (iii) we get

$$d(A_1u, A_iw) < M_{1i}(u,w) = d(A_1u, A_iw),$$

a contradiction. Hence $Su = A_1u = Tw = A_iw$ where $i>1$.

Since A_1, S and A_2, T are compatible mappings and since compatible maps commute at coincidence points, we get $A_1Su = SA_1u$ and

$A_2Tw=TA_2w$. Also, $A_1A_1u=A_1Su=SA_1u=SSu$ and $A_2A_2w=A_2Tw=TA_2w=TTu$. Then, by virtue of (ii), we get

$$d(A_1u,A_1A_1u) = d(A_1A_1u,A_2w) \leq hd(A_1A_1u,A_2w),$$

and $d(A_2A_2w,A_2w) = d(A_1u,A_2A_2w) \leq hd(A_1u,A_2A_2w)$,

This gives $A_1u=A_1A_1u=SA_1u$ and $A_2w=A_2A_2w=TA_2w$. Thus $A_1u= A_2w$ is a common fixed point of A_1, A_2, S and T. Moreover, if $A_2w \neq A_iA_2w$ for some $i>2$, using (iii) we have

$$d(A_1u,A_iA_2w) < d(A_1u,A_iA_2w),$$

a contradiction. Hence $A_1u = A_2w$ is a common fixed point of all the A_i, S and T. The proof is similar when it is assumed that every convergent S,T-iteration under A_1,A_2 converges to a point Tv in $SX \cup TX$. This establishes the theorem.

3. illustrative Examples. We now give an example to illustrate our theorem.

Example . Let $X = [2,\infty)$ with usual metric d. Define mappings A_i, S and $T: X \rightarrow X$, $i=1,2,3, \ldots$, by

$Sx = x+8$ when $x>2$, $\quad Sx = 2$ when $x=2$,

$Tx = 2x$ when $x \geq 3$, $\quad Tx = 2$ when $x < 3$,

$A_1x = A_2x=2$ for all x

and for $i > 2$.

$A_ix = 2(3+(1/i))$ when $x > 3+(1/i)$, $A_ix=2$ when $x \leq 3+(1/i)$.

Then $\{A_i\}$, S and T satisfy all the conditions of our theorem and have a unique common fixed point $x=2$.

In view of the above example, we now compare our theorem with some well known results.

I. Our result is more general than Theorem 5.1 of Jachynski [4]. Jachynski's theorem that $A_iX \subset SX$ and A_i to be compatible with T for each $i>2$. These conditions are neither required in our theorem nor satisfied in the example. In the above example it is obvious that A_iX is not contained in SX when $i>2$. To see that A_i and T are noncompatible when $i>2$, let us consider a decreasing sequence $\{x_n\}$ in $X=[2,\infty]$ such that $X_n \rightarrow 3+(1/i)$. Then $A_iX_n \rightarrow 2(3+(1/i))$ and $Tx_n \rightarrow 2(3+(1/i))$. But $A_iTx_n \rightarrow 2(3+(1/i))$ while $TA_iX_n \rightarrow 4(3+(1/i))$.Thus A_i and T are noncompatible when $i>2$. Moreover, Jachymski's theorem requires A_1,A_i, $i>2$, to satisfy the contractive condition $d(A_1x,A_iy) \leq \phi_i(m_{1i}(x,y))$ where $\phi_i:R_+ \rightarrow R_+$ is an upper semi continuous function such that $\phi_i(t)<t$ for

each $t>0$. This condition is not satisfied in the above example since the required function ϕ_i would not be upper semicontinuous at $t = 4+(2/i)$.

More importantly, our assumption that every convergent S,T-iteration under A_1,A_2 converges to some point in $SX \cup TX$ is strictly weaker than Jachymski's assumption that S or T be continuous. This is obvious because continuity of S or T together with compatibility of A_1,S and A_2,T implies our condition. However, in the above example neither S nor T is continuous. In this respect our threorem generalizes Jachynski's Theorem 3.3 also.

II. Our result is more general than that due to Rhoades, Park and Moon [15] in following respects. The theorem of Rhoades et al requires $A_iX \subset SX \cap TX$ and A_i to be compatible with both S and T for every value of $i>2$. These conditions are not required in our theorem and are not satisfied in the above example. Moreover, their result requires each A_i,A_j to satisfy an (ε,δ) type contractive condition (the necessary correction in their (ε,δ) condition appeared in Jungck et al [7]). However, in the above example, A_1,A_i, $i>2$, fail to satisfy (ε,δ) condition at $\varepsilon=4+(2/i)$.

III. Likewise, the present theorem is more general than theorem 2 of Pant, Joshi and Pande [11] in one respec. In Theorem 2 of Pant et al if we take $\{P_i\} = \{A_1,A_1,A_1, \ldots\}$ and $\{Q_i\} = \{A_2,A_3,A_4, \ldots\}$ then A_1,A_i are required to satisfy an (ε,δ) contractive condition for each $i>2$. However, as discussed above, this condition is not satisfied in the above example at $\varepsilon=4+(2/i)$.

It is clear form the above discussion that our theorem applies to a much wider class of mappings than covered by the above mentioned results in [4], [11] and [15]. Among the results which can either be obtained as special cases of our theorem or can be generalized in a straight forward manner in the light of our theorem we mention those due to Boyd and Wong [1], Das and Naik [2], Fisher [3], Joshi and Pant [5], Meir and Keeler [8], Pant [9,10], Park and Bae [12], Park and Rhoades [13], Rao and Rao [14], Sessa [16] and Tivari and Singh [17].

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