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# GRAPHS OF MORPHISMS OF GRAPHS

By

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**1. Introduction.** The aim of this paper is to develop and advertise categorical methods for the study of graphs. The main aim is the study of the graph of morphisms. That is, not only do we define morphisms of graphs, but we also construct for directed graphs $B$, $C$ *a morphism graph*, $D = DGRPH\,(B,C)$ whose vertices are the morphisms $B \rightarrow C$ of graphs, Thus in the graph $D$ we know the adjacency of morphisms.

These methods were used in [1] to study the symmetry of directed graphs. We will give here an outline of the methods for a more combinatorially minded readership, and will also give more examples of morphism graphs. We will also explain and develop some of the work of [5] on udirected graphs.

We first explain the background to our approach.

The spirit of categorical methods in graph theory, as in other areas, is to study a given graph, or a given construction on graphs, by means of its relations to all other graphs. The first step in this process is the notion of morphism between graphs, and so of the category of graphs which is to be studied. It is that part which is not usually emphasised in graph theory. The next part is the study of categorical constructions in the given category of graphs.

A second advantage of this method is that it allows a study of the relations between similar classes of objects, for example between various types of graphs, by the use of functors between categories, and of natural transformations between functors, This analysis of two categories of directed graphs is carried out in [1], from the point of view of topos theory.

Part of the basis of the methods used here comes from studies of the categorical foundations of the theory of sets, It was Lawvere who suggested that the 'proper' study of set theory was of the 'algebraic' and structural properties of the category of sets and functions, with a view to an axiomatisation of this category of sets and functions, with a view to an aximatisation of this category. This study was presumed to develop by studying the relations between defferent models of set theory, by comparison functors between different possible 'categories of sets'. Thus the aim of these studies is relativist, in a similar spirit to that of non-Euclidean geometry. There is no one 'category of sets', and there is the possibility of choosing whichever is convenient to the occasion.

These studies of the possible categories of sets have also led to analogies between sets and other areas of mathematics, by means of a study of the relations between categories. This is done by the use of functors between categories, and of natural transformations between functors.

A category of graphs may have many properties analogous to those of the category of sets, and this allows for new analogies. This approach is pursued with vigour in the papers [1], [2]. The work of [4], [5] has a different aim of studying the symmetry of a graph. From the categorical viewpoint, a symmetry group of a graph should be an object of the category of graphs, and so should be both a group and a graph. In particular, not only does the set of automorphisms have a composition, and so a group structure, but it also has an adjacency structure, so that one knows for instance when an automorphism is adjacent to the identity automorphism. Such automorphism are called *inner automorphisms* in [5]. These considerations also led to the notion of the *centre* of a directed graph, which in the finite case was found to be a direct sum of cyclic groups of order 2.

**1. The category $DG$ of directed graphs.** We consider a *directed graph* $A$ to be a set $E(A)$ of arrows and two functions $s,t : E(A) \to E(A)$, called the source and target functions repectively, such that $st = t$, $ts=s$. From this follows that $s^2 = s$, $t^2=t$, and $s$ and $t$ agree on $Im(s) = Im(t)$. We commonly write $x : sx \to tx$. An edge $x$ of $A$ is called a *loop* if $s(x) = t(x)$. The set $Im(s)$ is called the set of *verrices* of $A$ and is written $V(A)$. A vertex is usually denoted by a blob • and is also a loop, so that in this approach, each vertex of $A$ is identified with an associated loop. This is not the only approach to directed graphs. but it has certain advantages in terms of the properties of the category of graphs. These directed graphs are called *reflexive* in [1], [3].

The simplest directed graph is the empty graph $\phi$, and the next simplest is the *terminal* graph *T*, often written • with only one arrow, which is necessarily a vertex. There is a standard directed graph *I* which has three edges, two of them loops, and pictured as

$$0 = s(l) \bullet \xrightarrow{\ l\ } \bullet\ I = t(l)$$

A *morphism* $f : A \to A'$ of directed graphs is a function on the underlying sets which commutes with the source and target maps: $fs = sf$, $ft = tf$. So *f* maps loops to loops, and vertices to vertices. The point of this definition is that it allows morphisms of graphs which identify an arrow to a vertex.

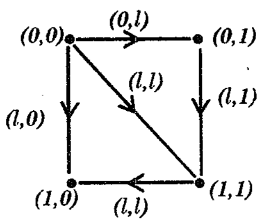
We now have a category of *DG* of directed graphs and their morphisms. The set of graph morphisms $A \to A'$ is written $DG\ (A,A')$. In particular, for an *A*, $DG\ (\phi,A)$ and $DG\ (A,T)$ are singletons, while $DG\ (T,A)$ is bijective with the set of vertices of *A*.

The graph *I* defined above plays a special role in the category of graphs, namely that the arrows of a graph *B* are bijective with the morphisms $I \to B$. Further, given two morphisms $f,g : B \to C$, we have $f = g$ if and only if $fx = gx$ for all morphisms $x : I \to B$. Thus *I* is a *generator* for the category *DG*. This property is analogous to the property of the group of integers in the category of groups. We can also determine the loops in a graph *C* as the morphisms $S \to C$ where *S* is the *loop* graph with one vertex *0* and one arrow $\sigma : 0 \to 0$.

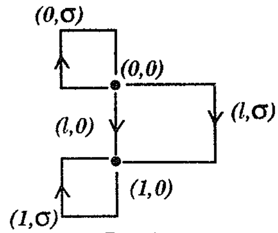
The main properties we will use for the category *DG* is the existence of products and of internal hom. The latter gives the graph of morphisms. We first define product.

A *product* of two graphs *A* and *B* consists of a graph *C* together with two morphisms $p : C \to A$, $q : C \to B$ with the property that if $f : D \to A$, $g : D \to B$ are two morphisms of graphs from the same graph *D*, then there is a unique graph morphism $h : D \to C$ such that $ph = f$, $qh = g$. Such a product is unique up to isomorphism, and is written $A \times B$.

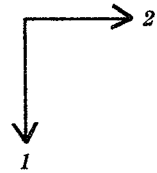
Note that we define the product by a universal property, which allows for the construction of morphisms. It is a consequence of the definition that the underlying set of $A \times B$ is the product of the underlying sets of *A* and of *B*, and the source and target maps are just the products of those of *A* and of *B*. So the pictures of $I \times I$ and $I \times S$ are as follows:




$I \times I$




$I \times S$




As another example, for any graph $A$ there is a canonical isomorphism $A \times T \to A$, where $T$ is the terminal graph.

The definition of the product is equivalent to the condition that the induced function of sets

$$(p_*, q_*) : DG\,(D,C) \to DG(D,A) \times DG\,(D,B)$$

is a bijection, where $p$ is the function $h \mapsto ph$, and similarly for $q_*$ Thus an internal property of the category of graphs, namely the existence of a product, is expressible in term of the basic category of abstract sets, with its usual cartesian product. The above bijection is of course analogous to the arithmetic law $(ab)^d = a^d b^d$.

Let $B$ and $C$ be graphs. The *internal hom* $DGRPH\,(B,C)$ is a graph defined by the condition that for all graphs $A$ there is a natural bijection

$$DG\,(A \times B\,,\, C) \equiv DG\,(ADGRPH\,(B,C)). \qquad (1.1)$$

This *exponential law* is analogous to the law $c^{ab} = (c^b)^a$. It corresponds to the usual notion that a function $f(a,b)$ of two variables $a$ and $b$ may also be thought of as a variable function of one variable:

$$a \to (b \to f(a,b)).$$

The exponential law (1.1) characterises the graph $DGRPH\,(B,C)$ up to isomorphism, and also allows us to prove its existence, by identifying its vertices with the morphisms $T \to DGRPH(B,C)$, or, equivalently, the morphisms $T \times B \to C$, and these are just the morphisms $B \to C$ and by identifying its arrows with the morphisms $I \to DGRPH\,(B,C)$. If $f,g$ are two morphisms $B \to C$, then the arrow $\psi\colon f \to g$ in $DGRPH\,(B,C)$ are the morphisms $\psi\colon I \times B \to C$ such that $\psi(0,-) = f$, $\psi(1,-) = g$. Thus $\psi$ is determined by assigning to each arrow $b : x \to y$ of $B$ an arrow $\psi(l,b) : fx \to gy$. So $\psi$ can be identified with a triple $(\psi' ; f,g)$ such that $\psi'$ is a function from the arrows of $B$ to those of $C$

and $s\psi' = fs$, $t\psi' = gt$. Either of these explicit descriptions can now be used as a definition of *DGRPH (B,C)*, and the exponential law (1.1) verified directly. As we shall explain later, the law is a special case of results on functor categories (see section 3), and so we do not give a proof here.

In the language of category theory, the exponential law says that the functor $- \times B$ is left adjoint to the functor *DGRPH (B, –)*, and we say that the category $\mathcal{DG}$ is *cartesian closed.*

There are number of useful properties of cartesian closed categories which we can exploit, and which can be interpreted in terms of our explicit description of the morphism graph. First, there is an evaluation morphism

$$\varepsilon_{B,C} : DGRPH\,(B\,,\,C) \times B \to C,$$

given by $(\psi,b) \to \psi\,(l,b)$. Second,there is a composition morphism of graphs

$$C_{A,B,C} : DGRPH\,(B\,,C) \times DGRPH\,(A,B) \to DGRPH\,(A,C)$$

in which the composite $\psi\phi$ of $\psi : I \times B \to C$ and $\phi : I \times A \to B$ is the comosite morphism

$$I \times A \xrightarrow{d \times 1} I \times I \times A \xrightarrow{1 \times \phi} I \times B \xrightarrow{\psi} C$$

where $d : I \to I \times 1$ is the diagonal morphism. Alternatively, in terms of the triple description, we have

$$(\psi'\,;\,f,g)\,(\phi'\,;\,h,\,k) = (\psi'\phi';\,fh,gk).$$

For a given graph $B$ we thus have the *endomorphism monoid* *END (B)*, which has the structure of both a graph and a monoid. That is, the monoid structure $END\,(B) \times END\,(B) \to END(B)$ is a morphism of graphs.

Any monoid in sets has a maximal subgroup. This is equally true in the category *DG* of directed graphs. This maximal subgroup of *END (B)* is written *SYM (B)* and called the *symmetry group-graph* of the graph $B$. in terms of the triple description, *SYM (B)* consists of triples $(\psi',f,g)$ as before such that each of $\psi'$, $f$, $g$ is a bijection. It is this group-graph which is studied in [4], [5]. Here we shall give more examples of morphism graphs, and shall examine an analogue for undirected graphs.

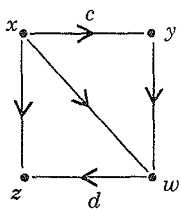
**2. Examples.** we now include a collection of examples which illustrate the main features of morphism digraphs. It is convenient to write $C^B$ for the graph *DGRPH (B,C)*.

1. For any graph $C$, there is a canonical isomorphism

$$C^T \to C,$$

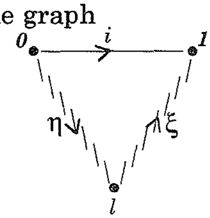
given by $(\psi\,; f,g) \mapsto \psi(0)$, where $0$ is the unique arrow of $T$.

2. For any graph $C$, we consider the graph $C^I$. The vertices of $C^I$ are bijective with the arrows of $C$. Suppose $c : x \to y$, $d: z \to w$ in $C$. Then an arrow $\psi: c \to d$ of $C^I$ is a morphism $\psi: I \times I \to C$ such that $\psi(0,l) = c$, $\psi(1,l) = d$, and so is specified by a diagram in $C$ of the type




Hence the number of arrows $c \to d$ in $C^I$ is the product of the cardinalities of $C(x,z)$, $C(x,w)$ and $C(y,w)$.

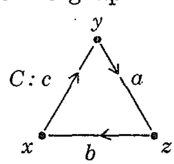
In particular, $I^I$ is the graph



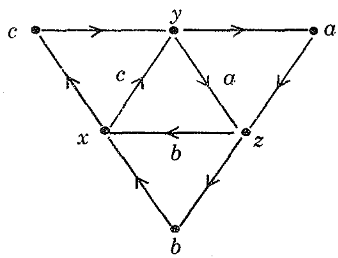

in which the arrows are defined uniquely by their end points. Note that there are two evaluation morphisms $\varepsilon_0 \varepsilon_1 : I^I \to I$, both of which map $l$ to $l$, while $\varepsilon_0$ maps $\eta \mapsto 0$, $\xi \mapsto l$, and $\varepsilon_1$ maps $\eta \mapsto 0$, $\xi \mapsto l$. Under the composition defined earlier, $I^I$ is a monoid graph in which $l$ acts as identity, $0$ and $1$ are left zeros,

3. More complicatedly, $S^I$ is the graph with two vertices and 8 arrows between any two of them, a total of 32 arrows.

4. Suppose now that $C$ is the graph

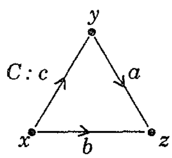



Then $C^I$ will be a graph with six vertices corresponding to the arrows of $C$ and will be of the following form:

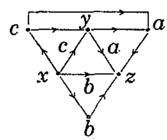



It is easy to show that, when $C$ is replaced by a directed cycle with $n \geq 4$ vertices, then $C^I$ is in effect obtained from $C$ by adding to each non-trivial arrow $a{:}x \rightarrow y$ of $C$ a pair of arrows $\eta_a : x \rightarrow a$, $\xi_a : a \rightarrow y$, so that $C^I$ contains $2n$ more arrows than does $C$.

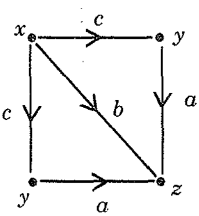
Suppose now that $C$ is the graph



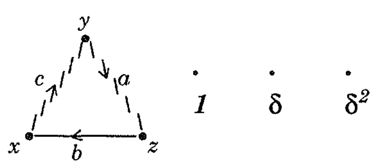

Then $C^I$ is of the following form




where the arrow $c \rightarrow a$ in $C^I$ is the morphism $I \times I \rightarrow C$ given by the diagram




5. Let $C$ be the first example in Example 4 and consider the graph $C^C$. The automorphisms of $C$ are the identity $1$ and the cyclic permutations $\delta$ and $\delta^2$. The only other endomorphisms of $C$ are constant. So $C^C$ is the graph.




and $SYM(C)$ is discrete.

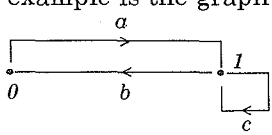
6. Denote by $K_{n,k}$ the *complete digraph* with $n$ vertices and, for each pair of vertices $u$ and $v$, exactly $k$ arrows with source $u$ and target $v$. By previous results, $(K_{n,k})^1$ has vertices the arrows of $K_{n,k}$ and has $k^3$ arrows between any two of them. That is, $(K_{n,k})^l$ is isomorphic to $K_{n^2k,k^3}$.

7. Consider $C = K_{n,k}$ , $k \geq 1$. An endomorphism $f : C \to C$ consists of an arbitrary functin on the vertex set of $C$ together with for each vertex $x$ a function $C(x,x) \to C(fx, fx)$ sending the arrow $x$ to the arrow $fx$ , and for each distirict pair of vertices $x$ , $y$ an arbitrary function $C(x,y) \to C(fx, fy)$. Hence $End(C)$ has $n^n k^{(k-1)n} k^{kn(n-1}$ elements. The arrows $f \to g$ are determined by $k$ choices of arrow $fx \to gy$ for each pair of vertices $x$ ,$y$ and each of the $k$ choices of arrow $c : x \to y$ in $C$, giving $k^{kn^2}$ arrows $f \to g$. Thus the graph $END(C)$ is also complete. In particular if $k = 1$, then the group $Aut(C)$ is isomorphic to the symmetric $S_n$, and the underlying graph of $SYM(\Delta_{n,1})$ is isomorphic to $K_{n!,1}$.

7. Let $C$ be the graph $K_{1,3}$, with vertex $0$ and two nontrivial loops $l$ and $m$. The automorphism group $Aut(C)$ is isomorphic to $z_2$ with non-trivial arrow the involution $\rho$ interchanging $l$ and $m$. Since $0$, $l$ and $m$ all have source and target $0$, the arrows of $SYM(C)$ are $(\psi;f,g)$ for all permutations $\psi$ of $\{0,l,m\}$ and every $f,g \in \{1,\rho\}$. Thus the group of $SYM(\Delta)$ is isomorphic to $S_3 \times C_2 \times C_2$ and every $\psi$ occurs four times. The graph $SYM(\Delta)$ is isomorphic to $K_{2,6}$ . There are seven endomorphisms of $\Delta$ other than $1$ and $\rho$, and so $END(C)$ is isomorphic to $K_{9,27}$.

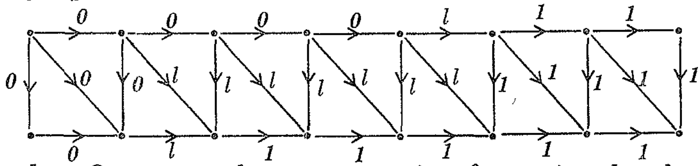
Similarly, when $C = K_{1,k}$, we have $SYM(C) \cong K_{(k-1)!,\, k!}$, with group isomorphic to $S_k \times S_{k-1} \times S_{k-1}$, and $END(C) \cong K_{k^{k-1}k^k}$.

8. An interesting example is the graph $\Omega$ as follows :




The morphisms $X : B \to \Omega$ may be identified with the subgraphs of $B$, by $\chi \to \chi^{-1}(1)$. The reason is that if $A$ is a subgraph of $B$, then it has a characteristic morphism $XA : B \to \Omega$ which maps only $A$ to $1$, maps the vertices of $B \setminus A$ to $0$, and maps the arrows of $B$ not in $A$ to the only possible choice. Because of this property, the graph $\Omega$ is called a *subobject classifier* for directed graphs. The lattice $DG(\Omega, \Omega)$ of subgraphs of $\Omega$ is not Boolean, since for example the complement of the complement of {1} is {*1*,*c*}. The existence of this example is, as we shall see in the next section, one of the reasons why directed graphs may be considered as analogous to sets.

9. Let $I_n$ be the directed graph with $n + 1$ vertices $0, 1, \ldots, n$, and one non-trivial arrow $l_i : i \to i+1$ for $0 \le i \le n-1$. Then there is a unique morphism $f_i : I_n \to I$ such that $l_i \mapsto l$. Further, in the graph $(I_n)^I$ there is a unique morphism $f_j \to f_i$ for each $j \ge i$. Thus the graph $(I_n)^I$ is a simplex with $n+1$ vertices. This is illustrated with the following diagram of an arrow $f_5 \to f_2$:




**Remark.** One can use known properties of cartesian closed categories to analyse other constructions and examples. For example, it is a property of cartesian closed categories that there is an isomorphism

$$(B \times C)^A \cong B^A \times C^A$$

More generally, if $C$ is given as a *limit* $C = lim_\lambda C_\lambda$, then $C^A$ is isomorphic to the limit $lim_\lambda (C_\lambda)^A$. In the other direction, if $A$ is given as a *colimit* $colim_\lambda A_\lambda$, then $B \times A$ is isomorphic to $colim_\lambda (B \times A_\lambda)$, and $C^A$ is isomorphic to $lim_\lambda (C^{A\lambda})$. These results can be practical tools for computation of examples of product and morphism graphs.

**3. The category *DG* is a topos.** A *topos* is a category with properties analogous to those of the category *sets* of sets and functions. Specifically, it has finite limits, is cartesian closed, and has a subobject classifier [4]. This last is a property analogous to the characteristic function of a subset of a set. We explain this here, since it exposes the difference between the *logic* of directed graphs and that of ordinary sets.

**4. Undirected graphs.** We now consider whether similar results, and in particular an endomorphism monoid-graph and symmetry group-graph, can be obtained in the undirected case.

It is convenient to consider an undirected graph $A$ as having a single set of elements, called edges, and to replace the source and target maps of the directed case by a single boundary or end-points map $\partial$. For $X$ a set we denote by $S^2(X)$ the set of unordered pairs $(x,y)$ of elements of $X$. Thus $S^2(X)$ can be regarded as the quotient fo $X \times X$ by the twist action of the cyclic group of order *2*, in which $(x,y) \to (y,x)$. We write $d(X)$ for the diagonal in $X \times X$ and for its image in $S^2(X)$. The elements of $d(X)$ are called singletons.

A *graph* $A$ is a set $A$ of *edges* together with a *boundary* map $\partial : A \to S^2(A)$ satisfying one contition. Let the set of *vertices* of $A$ be $V(A) = \{v \in A : \partial v = \{v\}\}$. We require that $\partial(A) \subseteq S^2(V(A))$. *A loop* of $A$ is any edge of $A$ whose boundary is a singleton. In particular, any vertex of $A$ is also a loop. The set of loops of $A$ is written $L(A)$. A *link is* an edge of $A$ which is not a loop.

A function $f : A \to B$ of undirected graphs is a graph *morphism* if

$$\partial f = f_* \partial : A \to S(V(B))$$

where $f_* : S^2(A) \to S^2(B)$ is defined by applying $f$ to each edge. This yields a category *Ugrph* of (undirected) graphs and graph morphisms.

Again, the *product* in *Ugrph* is defined by the usual universal property. Its existence is proved as follows by a direct construction. Let $A$ and $B$ be undirected graphs. Their product graph $A \times B$ has vertex set the product of the vertex sets of $A$ and $B$, while the edges of the product consist of all pairs

$$((x_1, x_2), \{(u_1, u_2), (v_1, v_2)\})$$

where $\partial x_1 = \{u_1, v_1\}$, $\partial x_2 = \{u_2, v_2\}$, and where

$$\partial((x_1, x_2), \{u_1, u_2), (v_1, v_2)\}) = \{(u_1, u_2), (v_1, v_2)\}.$$

The projections from the product to $A$ and $B$ send

$$((x_1, x_2), \{(u_1, u_2), (v_1, v_2)\})$$

to $x_1$ and $x_2$ repectively. This implies that if neither $x_1$ nor $x_2$ are loops, then the product contains two edges which project to $x_1$ and $x_2$. The universal property follows from the construction, because if $f_1, f_2$ are morphisms as above, then the morphism $f$ with components $f_1$ and $f_2$ is $(V(f_1), V(f_2))$ on vertices, and sends an edge $y$ with boundary $\{a,b\}$ to the edge of the product

$$((f_1 y, f_2 y), \{(f_1 a, f_2 b) \; (f_1 b, f_2 a)\}).$$

This the only definition consistent with $f$ being a morphism with the required projections. Of course the other edge of the product

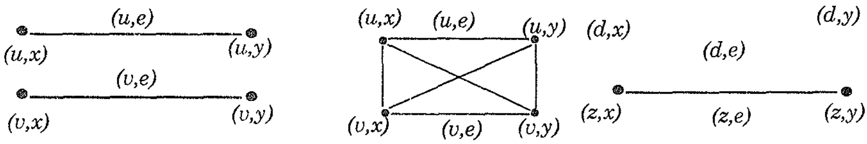
$$((f_1 y, f_2 y), \{(f_1 a, f_2 b) \; (f_1 b, f_2 a)\}).$$

also projects to $f_1 y$ and $f_2 y$, but it is not in general the value of a morphism on $y$.

The category *Ugrph* is *not* cartesian closed, and hence cannot be expressed in the form $set^{B^{op}}$ , To demonstrate this we show in the following counterexample that products do not preserve coequalisers, [4,p.44]. For the categorical background of this example, we refer to [6].

**4.1 Counterexample.** Let $A$ be the discrete graph on two vertices $x$ , $y$ and let $B$ be the graph with two vertices $u$ , $v$ and a single edge $e$ between them. Define $f$ , $g : A \rightarrow B$ by $f$ takes $x$ , $y$ to $u$ , $v$ respectively, while $g$ maps $x$ and $y$ to $u$. As coequalieser of $f$ and $g$ we can take the morphism $h : B \rightarrow C$ where $C$ is the graph with one vertex $z$ and one non-trivial loop $d$ *at* $z$. The morphism $h$ map $e$ to $d$.

Consider now the graphs $A \times B$, $B \times B$, $C \times B$ .




$A \times B$ $B \times B$ $C \times B$

The coequaliser of $f \times 1$, $g \times 1 : A \times B \rightarrow B \times B$ is pictured as

and has one more edge than $C \times B$ .

**5. Involutary Digraphs.** One advantage of the formulation *DG* = $let^{6^{op}}$ is that the endomorphism graph $A^A$ of endomorphisms of $A$ is a monoid object in the category *DG* and so is also a digraph. This construction is not available for undirected graphs. One solution to this difficulty is to embed *ugrph* in the subcategory *IDG*. of *involutary digraphs* in *DG*. Thus to each undirected graph $A$ we wish to associate an involutary directed graph $B$ having endomorphism monoid isomorphic to that of $A$, calculate the involutary directed graph $B^B$ and map back to a graph *END(A)*.

An *involution* $r : A \rightarrow A$ of a digraph $A$ is a function satisfying $r^2 = id$, $sr = t$, $tr = s$, $rt = t$. We call $rx$ the *reverse* of $x$. Note that $rv = v$ for every vertex $v$. A loop $x$ of $A$ is called *self-involutary* if $rx = x$. An *involutary digraph* is a pair $(A,r)$ where $r$ is an involution of $A$. A morphism $f: (A,r) \rightarrow (B, r')$ of involutary digraphas is a graph morphism $f: A \rightarrow B$ which satisfies $fr = r'f$. This gives a category $\mathcal{IDG}$ of involutary digraphs.

We now wish to construct a functor from *ugrph* to *IDG*. This should map vertices to vertices and edges to involutary pairs of edges, but there is a choice for loops. Each loop may be mapped either to a self-involutary loop or to an involutary pair of loops. In the following definition we choose to map every loop to a self-involutary loop. This is convenient for what follows.

**5.1 Definition.** The *doubling functor* $D$ : *ugrph* $\rightarrow$ *IDG* maps a graph $A$ to an involutory digraph $\mathcal{D}(A) = (B,r)$, having exactly the same vertices and loops but twice the number of edges. So if $e$ is an edge of $A$ with vertices $x$ and $y$, there are in $\mathcal{D}(A)$ edges $(e,x,y) : x \rightarrow y$ and $(e,y,x)$ $: y \rightarrow x$, which coincide if $x = y$. The involution is given by $r(e, x, y) = (e, y, x)$.

*If* $f : A \rightarrow B$ is a morphism of undirected graphs, then $\mathcal{D}(f) : \mathcal{D}(A) \rightarrow \mathcal{D}(B)$ is defined by

$$D(f)(e,x,y) = (fe, fx, fy).$$

**5.2 Difinition** The *halving functor* $\mathcal{H} : \mathcal{IDG} \rightarrow$ *ugrph* maps $(B,r)$ to an undirected graph $\mathcal{H}(B,r) = A$ with the same vertices as $B$, and with edges the unordered pairs $\{e,re\}$, taken to have vertices the unordered pair $\{se,te\}$.

Denote by $\mathcal{SIDG}$ the category of involutary digraphs with self-involutary loops. This is the image of *ugrph* under $\mathcal{D}$.

**Proposition 5.3.** *The functors*

$$\mathcal{D} : ugrph \rightarrow \mathcal{SIDG} \quad \mathcal{H} : \mathcal{SIDG} \rightarrow ugrph$$

*yield an equivalene of categories.*

**Proof.** If $A$ is an undirected graph, then an isomorphism is given on edges by $\{(e,x,y), (e,y,x)\} \rightarrow e$. If $(B,r)$ is an involutary digraph with self involutary loops, then an isomorphism $\mathcal{DH}(B,r) \rightarrow (B,r)$ is given by $(\{e, d\}, x, y) \rightarrow e$ whenever $e : x \rightarrow y$ in $B$.

As explained above, the category *IDG* is a category of action of a monoid, and so is a topos and is particular is caresian closed and has an internal endomorphism monoid for any of the object.

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