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A NOTE ON MAXIMAL SUBGROUPS OF $Sp_4(q)$ $-q$ ODD

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ABSTRACT

The aim of this paper is to give a very simple description to the maximal subgroups of $Sp_4(q)$, $q = p^a$, a is a power of an odd prirme.

We first explain the background to our approach.

1. Introduction. Subgroups of quaternary abelian linear groups ware studied by Dickson [3], where the complicated notion of substitution was used. The term quaternary abelian linear groups was used in place of "symplectic groups". Weyl has also used the term "complex group". The maximal subgroups of the projective symplectic group $PSP_4(q)$, q odd were determined by H.H. Mitchell [4]. The object of this paper is to give a very simple description to maximal subgroups using elementary algebra notions.

2. Basic definitions .

Definition 2.1. The symplectic group $Sp_4(q)$ is the set of all 4×4 matrices X over the finite field $Gf(q)$, which satisfy the equation

$X^t px = X$ ... (1)

where

$$P = \left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right]$$

and X^t denotes the transpose matrix of X.

Definition 2.2 . A set $| e_1, e_2, f_1, f_2 | \subseteq V(4,q)$, where $V(4,q)$ is a 4 dimensional vector space $\{(x, y, z, t) : x, y, t \in Gf(q)\}$, is a symplectic base if

$$(e_i, f_j) = \delta_{ij},\ (f_j, e_i) = -\delta_{ij},\ \text{where } \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

and (,) is an alternating bilinear form on $V(4,q)$. For more details see [1].

Note. If K, H are two subgroups of G , K G, then $C = K:H$ will denote the semidirect product. of K by H.

3. Results.

Lemma 3.1. If $g \in \langle e_1, f_1\rangle^{Sp_4(q)}$ (the stabilizer of $\langle e_1, f_1\rangle$ in $Sp_4(q)$) then g stabilizes $\langle e_1, f_1\rangle$.

Proof. Let $e_1^g = ae_1 + bf_1$ and $f_1^g = ce_1 + df_1$ then
$(e_1, f_1) = (e_1^g, f_1^g) = (ae_1 + bf_1, ce_1 + df_1) = ad - bc = 1$
Let $f_2^g = xe_1 + \alpha e_2 + yf_1 + \beta f_2$ and $e_2^g = y'e_1 + \gamma e_2 + x'f_1 + \delta f_2$
then $(e_2^g, f_1^g) = 0 = y'd - x'c$
and $(e_2^g, e_1^g) = 0 = y'b - x'a$
solving these two equations simultaneousely one has $(ad-bc)x' = 0$ which implies that $x' = 0$.
Similarly we prove that $x = y = y' = x' = 0$, this means that

$$f_2^g = \alpha e_2 + \beta f_2 \text{ and } e_2^g = \gamma e_2 + \delta f_2.$$

Therefore g stabilizes $\langle e_2, f_2\rangle = \langle e_1, f_1\rangle^+$ the orthogonal space to $\langle e_1, f_1\rangle$.

Theorem 3.2 If $\Gamma = SL(2,q) \times \langle t\rangle$ is the Wreath product of SL (2,q) by $\langle t\rangle$ the cyclic group generated by $t = \left[\begin{array}{c|c} O & I \\ \hline I & O \end{array}\right]$
then $\langle e_1, f_1\rangle^{Sp_4(q)} \cong \Gamma$.

Proof. From lemma 3.1, one obtains

$$<e_1, f_1>^{Sp_4(q)} = \{<e_1, f_1>, <e_2, f_2>\}^{Sp_4(q)} = \left\{\left[\begin{array}{c|c} g & \\ \hline & h \end{array}\right], \left[\begin{array}{c|c} & g' \\ \hline h' & \end{array}\right]\right\}$$

where $g, h, g', h' \in SL(2.q)$.

Let H be the semidirect product of $SL(2.q) \times SL(2.q)$ by z_2, then the map $\sigma: H \to \Gamma$ defined by $\sigma((g,h), t) = \left[\begin{array}{c|c} g & \\ \hline & h \end{array}\right] t$, is an isomorphism for : $\sigma((g, h), t)((g_1, h_1), 1) = \sigma((g, h), t)((g_1, h_1)^t, 1) = \sigma((gh_1, hg_1), t) =$

$$\left[\begin{array}{c|c} & gh_1 \\ \hline hg_1 & \end{array}\right] = \left[\begin{array}{c|c} & g \\ \hline h & \end{array}\right] \left[\begin{array}{c|c} g_1 & \\ \hline & h_1 \end{array}\right] = \sigma((g, h), t)\ \sigma((g_1, h_1), 1)$$

and σ is one to one. Therefore σ is an isomorphism and $H \cong \Gamma$

Theorem 3.3. $<e_1, e_2>^{Sp_4(q)} \cong E : GL(2,q)$, where E is an abelian group of order q^3

Proof. It is obvious that

$$<e_1, e_2>^{Sp_4(q)} = \left[\begin{array}{cc|cc} a & b & \multicolumn{2}{c}{O} \\ c & d & & \\ \hline \alpha & \beta & a' & b' \\ \gamma & \delta & c' & d' \end{array}\right]$$

where $e_1^g = ae_1 + be_2$, $e_2^g = ce_1 + de_2$,

$g \in \langle e_1, e_2 \rangle^{Sp_4(q)}$, $f_1^g = \alpha\, e_1 + \beta\, e_2 + a' f_1 + b' f_2$ and

$f_2^g = \gamma\, e_1 + \delta\, e_2 + c' f_1 + d' f_2$. Also one has

$$1 = (e_1^g, f_1^g) = (ae_1 + be_2\,,\ \alpha\, e_1 + \beta\, e_2 + + a' f_1 + b' f_2) = a a' + b b' \qquad \dots (2)$$

$$0 = (e_1^g, f_2^g) = (ae_1 + be_2\,,\ \gamma\, e_1 + \delta\, e_2 + + c' f_1 + d' f_2) = a c' + b d' \qquad \dots (3)$$

$$0 = (e_2^g, f_2^g) = (ce_1 + de_2\,,\ \alpha\, e_1 + \beta\, e_2 + + a' f_1 + b' f_2) = c\, a + db \qquad \dots (4)$$

$$1 = (e_2^g, f_2^g) = c c' + d d' \qquad \dots (5)$$

From equations (2) , (3), (4) and (5) one obtains

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a' & c' \\ b' & d' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus $\langle e_1, e_2 \rangle^{Sp_4(q)} = \left\{ \left[\begin{array}{cc|c} \multicolumn{2}{c|}{g} & O \\ \hline \alpha & \beta & \multirow{2}{*}{$(g^{-1})^t$} \\ \gamma & \delta & \end{array} \right] \ g \in GL(2,q) \right\}$.

Let $\left[\begin{array}{cc|cc} 1 & 0 & \multicolumn{2}{c}{O} \\ 0 & 1 & & \\ \hline \alpha & \beta & 1 & 0 \\ \gamma & \delta & 0 & 1 \end{array} \right] \in \langle e_1, e_2 \rangle^{Sp_4(q)}$, then it follows that

$0 = (\alpha e_1 + \beta e_2 + f_1\,,\ \gamma\, e_1 + \delta\, e_2 + f_2)$, this means that $\beta = \gamma$ and therefore

$\langle e_1, e_2 \rangle^{Sp_4(q)} = \left\{ \left[\begin{array}{c|c} g & O \\ \hline A & (g^t)^{-1} \end{array} \right], g \in GL(2,q) \right\}$ where A is 2×2 symmetric matrix.

Using (1), one can show that if $\left[\begin{array}{c|c} g & O \\ \hline A & (g^t)^{-1} \end{array} \right] \in \langle e_1, e_2 \rangle^{Sp_4(q)}$

then $Ag^t = gA^t = gA$, this means that Ag is symmetric.

So $\langle e_1, e_2 \rangle^{Sp_4(q)}$ is isomorphic to $M = \left\{ \left[\begin{array}{c|c} g & O \\ \hline Ag & (g^t)^{-1} \end{array} \right] \right\}$.

Now let $E = \left\{ \left[\begin{array}{c|c} I & O \\ \hline A & I \end{array} \right] \right\}$ and $G = \left\{ \left[\begin{array}{c|c} g & \\ \hline & (g^t)^{-1} \end{array} \right] \right\}$ and let $E : G$

be the semidirected product of E by $GL(2,q)$, then the map $\rho : E : G \to M$ defined by

$\rho\,(A\,,\,g) = \left[\begin{array}{c|c} g & \\ \hline Ag & (g^t)^{-1} \end{array} \right]$ is an isomorphism for

$\rho((A,g)(B,h)) = (A+B^g, gh)$ where (B,h), $(A,g) \in E$ and $B^g = (g^t)^{-1} B_g$.

So $\rho((A,g)(B,h)) = \left[\begin{array}{c|c} gh & O \\ \hline Agh + (g^t)^{-1} Bh & ((gh)^t)^{-1} \end{array}\right] =$

$\rho(A,g)\,\rho(B,h)$ and ρ is one to one. Therefore $H \cong E{:}G$.

Theorem 3.4 . The stablizer of a one dimensional subspace $< f_1 >$ is isomorphic to the semidirect product $N : SL(2,q) \times z_{q-1}$

where $N = \left\{\begin{bmatrix} 1 & a & b & c \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{bmatrix}\right\}$

Proof . Let $g \in Sp_4(q)$ such that $f_1^g = \lambda^{-1} f_1$ and

$e_1^g = xe_1 + ae_2 + bf_2 + cf_1$, $e_2^g = x'e_1 + \alpha e_2 + f_2 + zf_1$

$f_2^g = x''e_1 + \gamma e_2 + \delta f_2 + z' f_1$ then using the relations

$(e_1^g, f_1^g) = (e_2^g, f_2^g) = 1$ and $(e_1^g, e_2^g) = (f_1^g, f_2^g) = 0$

one obtains

$$< f_1 >^{Sp_4(q)} = \left\{\begin{bmatrix} \lambda & a & b & c \\ 0 & \alpha & \beta & \lambda(\alpha b - a\beta) \\ 0 & \gamma & \delta & \lambda(b\gamma - a\delta) \\ 0 & 0 & 0 & \lambda^{-1} \end{bmatrix}\right\}$$

where $\alpha\delta - \beta\gamma = 1$, For details see [2]

Let $e = \begin{bmatrix} 1 & v_1 & v_2 & t \\ 0 & 1 & 0 & v_2 \\ 0 & 0 & 1 & -v_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in N$ and indentify e with (t,v)

where $v = (v_1, v_2) \in V(2,q)$, then one has

$(t,v)(s,w) = (s+t+(v/w), v+w)$ where $(v/w) = v_1w_2 - v_2w_1$ and

$$(t,v)^g = \begin{bmatrix} 1 & \alpha v_1 + \gamma v_2 & \beta v_1 + \delta v_2 & t \\ 0 & 1 & 0 & \delta v_2 + \beta v_1 \\ 0 & 0 & 1 & -\gamma v_2 + \alpha v_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = (t, vg),$$

where $g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL(2,q)$. Also if $a = \begin{bmatrix} \lambda_1 & \\ & 1_\lambda \end{bmatrix} \in z_{q-1}$, then

$(t,v)^a = (\lambda^2 t, \lambda v)$. without loss of generality let $\lambda = 1$ and let Γ be the semidirect product of N by $SL(2,q)$ i.e Γ is the set of all pairs $(g : (t,b))$, then the map

$\psi : \Gamma \to \left\{\begin{bmatrix} 1 & v_1 & v_1 & t \\ 0 & \alpha & \beta & \alpha v_2 - \beta v_1 \\ 0 & \gamma & \delta & \gamma v_2 - \delta v_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right\}$ defined by

$$\psi\,(g:(t,b)) = \begin{bmatrix} 1 & v_1 & v_1 & t \\ 0 & \alpha & \beta & v_2\alpha-\beta v_1 \\ 0 & \gamma & \delta & \gamma v_2-\delta v_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ is an isomorphism}$$

for, $\psi\,(g:(t,b))\,(h:(s,w)) = \psi\,(gh:(t,v)^h(s,w)) =$
$\psi\,(gh:(t,v^h)(s,w)) = \psi\,(gh:(t+s(vh/w),\, vh+w)) =$
$\psi\,(gh:(t+s+\acute{\alpha}v_1w_2+\acute{\gamma}v_2w_2-\beta'v_1w_1-\delta'v_2w_2,\; w_1+\acute{\alpha}v_1+\acute{\gamma}v_2,\; w_2+\acute{\beta}v_1+\acute{\delta}v_2$
$) =$

$$\begin{bmatrix} 1 & v_1 & v_2 & t \\ 0 & \alpha & \beta & \alpha v_2-\beta v_1 \\ 0 & \gamma & \delta & \gamma v_2-\delta v_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & w_1 & w_2 & t \\ 0 & \acute{\alpha} & \beta' & \acute{\alpha}w_2-\beta'w_1 \\ 0 & \acute{\gamma} & \delta' & \acute{\gamma}w_2-\delta'w_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$= \psi\,(g:(t,b))\,\psi\,(h:(s,w))$

Therefore ψ is a homorphism, furthermore ker $\psi = \{1, (0,0)\}$

Thus ψ is an isomorphism and hence

$$<f_1>^{Sp_4(q)} \cong N{:}SL\,(2,q)\times z_{q-1}.$$

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