Jñānābha, Vol. 28, 1998

SOME FIXED POINT THEOREMS IN HAUSDORFF SPACES

By

M.S. Rathore

Department of Mathematics

Govt. Postgraduate College, Sehore, M.P., India

and

Ravindra Garg*

Department of Mathematics

Institute for Excellence in Higher Education, Old Walmi Campus,

Kolar Road, P.B.No.– 588, Bhopal– 462016, M.P., India

(Received : May 20, 1997)

ABSTRACT

In this paper, we shall prove some fixed point theorems in Hausdorff spaces. Our result is an improvement of the results of many authors e.g. Popa [4], Thomas [5] and Chugh and Rani [2].

Introduction. Several authors such as Wong [6], Chatterjee and Ghosal [1], Fisher and Khan [3], Popa [4], Thomas [5], Chug and Rani [2] etc. have obtained a number of intresting results on fixed point for different types of contractive mappings in Hausdorff spaces. The object of this paper is to establish some fixed point theorems in Housdorff spaces which are generalizations of results due to [4], [5] and [2].

Preliminaries. Let R^+ denote the set of non-negative reals and $w : R^+ \to R^+$ is a continuous function and non decreasing in each coordinate variable such that $0 < w(t) < t$ for all $t > 0$.

Main Results.

Theorem 1. Let T be a continuous mapping of a Hausdorff space X into itself. Let f be a continuous mapping of $X \times X$ into R^+ such that

(1.1) $f(x,y) \neq 0$ for all $x \neq y \in X$

(1.2) $f(x,y) \geq f(x,x)$ and $f(x,y) \geq f(y,y)$ for all $x \neq y \in X$

(1.3) $f(Tx,Ty) \leq w(\max\{f(x,y), f(x, Tx), f(y, Ty), f(y, Tx), f(Ty,T^2x)\})$ for all $x \neq y \in X$.

If for same $x_0 \in X$ the sequence $\{x_n\} = \{T^n x_0\}$ has a convergent

* *Address for Correspondence* : Professor Ravindra Garg, F-116/22, Shivaji Nagar, Opposite Ankur Complex, Near Bus stop No. 6½, Bhopal -462016, M.P., India

subsequence, then T has a unique fixed point.

Proof. By (1.3), we have

$f(x_1,x_2) = f(Tx_0, Tx_1)$

$\leq w(max\{f(x_0,x_1), f(x_0,Tx_0), f(x_1, Tx_1), f(x_1, Tx_0), f(Tx_1, T^2x_0)\})$

$= w(max\{f(x_0,x_1), f(x_1, x_2), f(x_1, x_1), f(x_2,x_2)\})$

$= w(max\{f(x_0,x_1), f(x_1, x_2)\})$. (By condition (1.2)).

If $f(x_0,x_1) < f(x_1,x_2)$, then $f(x_1,x_2) \leq \omega(f(x_1,x_2)) < f(x_1,x_2)$, a contradiction. Thus $f(x_1,x_2) \leq f(x_0,x_1)$.

Repeating the above argument, we have

$f(x_0,x_1) \geq f(x_1,x_2) \geq f(x_2,x_3) \geq \ldots$

This shows that $\{f(x_n,x_{n+1})\}$ is the monotonic decreasing sequence of positive real numbers which must converge with all its subsequence to some u,u being real. Let $\{x_n\}$ has a convergent sub-sequence $\{x_{n_k}\}$ which converges to some $x \in X$.

By the continuity of T, we have

$Tx = T(lim\ x_{n_k}) = lim\ (Tx_{n_k}) = lim\ x_{n_k+1}$,

$T^2x = T(lim x_{n_k+1}) = lim\ (Tx_{n_k+1}) = lim\ x_{n_k+2}$.

Now we prove that x is a fixed point of T.We have

$f(x,Tx) = f(lim\ x_{n_k}, lim\ x_{n_k+1})$

$= lim\ f(x_{n_k},x_{n_k+1})$

$= u = lim\ f(x_{n_k+1},x_{n_k+2}) = f(Tx, T^2x)$.

If $x \neq Tx$,then condition (1.3) gives

$f(Tx, T^2x) \leq w(max\{f(x,Tx), f(x, Tx), f(Tx, T^2x), f(Tx, Tx), f(T^2x, T^2x)\})$,

$f(Tx, T^2x) = f(x,Tx) \leq w(f(x, Tx)) < f(x, Tx)$ (as above).

This is a contradiction. Therefore $x = Tx$.

To prove uniqueness, let $y \neq x$ be another fixed point of T, then

$f(x, y) = f(Tx, Ty)$

$\leq w(max\{f(x,y), f(x, Tx), f(y, Ty), f(y, Tx), f(Ty, T^2x)\})$

$= w(max\{f(x,y), f(x, x), f(y, y), f(y,x), f(y, x)\})$

$= w(max\{f(x,y), f(x, x), f(y, y)\})$.

By condition (1.2), we have

$max\{f(x,y), f(x,x), f(y,y)\} = f(x,y)$. Hence we have

$f(x,y) \leq w(fx,y)) < f(x,y)$

giving a contradiction. Hence $x = y$. Thus the fixed point is unique.

This completes the proof of the theorem.

Now we establish the following theorem for a pair of continuous mappings in a Hausdorff Space.

Theorem 2. Let T_1 and T_2 be continuous mappings of a Hausdorff space X into itself. Let f be a symmetric continuous mapping of $X\times X$ into R^+ satisfying (1.1) , (1.2) and

(2.1) $f(T_1x, T_2y) \leq w(max\{f(x,y), f(x, T_1x), f(y, T_2y), f(y, T_1x), f(T_2y, T_2T_1x)\})$

for all $x \neq y \in X$.

If for some $x_0 \in X$, the sequence $\{x_n\}$ where $T_1x_{2n} = x_{2n+1}$ and $T_2x_{2n+1} = x_{2n+2}$ for $n = 0,1,2,3, ...$, has a convergent subsequence $\{x_{(2p+1)n}\}$, where $p \in N$ is fixed and, $n \in N$, then T_1 and T_2 have a unique common fixed point.

Proof. By applying (2.1), for $n=0,1,2,3 ,....,$

$f(x_1,x_2) = f(T_1x_0, T_2x_1)$

$\leq w(max\{f(x_0,x_1), f(x_0,T_1x_0), f(x_1, T_2x_1), f(x_1, T_1x_0), f(T_2x_1,T_2T_1x_0)\})$

$= w(max\{f(x_0,x_1), f(x_0,x_1), f(x_1, x_2), f(x_1, x_1) f(x_2, x_2)\})$,

which implies (as in the proof of theorem1)

$$f(x_0,x_1) \geq f(x_1,x_2)$$

and repeating the above argument, we have

$$f(x_0,x_1) \geq f(x_1,x_2) \geq f(x_2,x_3) \geq ...$$

and thus the sequence $\{f(x_n, x_{n+1})\}$ converges to some $u \in R$. Let $\{x_n\}$ has a convergent subsequence $\{x_{(2p+1)n}\}$ converges to some $x \in X$. Let $\{x_{(2p+1)2n}\}$ be a subsequence of $\{x_{(2n+1)n}\}$.

By continuity of T_1 and T_2, we get

$T_1x = T_1(lim x_{(2p+1)2n}) = lim\ T_1x_{(2p+1)2n} = lim\ x_{(2p+1)2n+1}$,

$T_2T_1x = T_2(lim\ x_{(2p+1)2n+1}) = lim\ (T_2x_{(2p+1)2n+1}) = lim\ (x_{(2p+1)2n+2})$,

$f(T_1x,T_2T_1x) = f(lim\ x_{(2p+1)2n+1}, lim\ x_{(2p+1)2n+2})$

$= lim\ f(x_{(2p+1)2n+1}, x_{(2p+1)2n+2})$

$= u = lim\ f(x_{(2p+1)2n}, x_{(2p+1)2n+1})$

$= f(lim x_{(2p+1)2n}, lim x_{(2p+1)2n+1}) = f(x, T_1x)$

If $x \neq T_1x$,

$f(x, T_1x) = f(T_1x, T_2T_1x)$

$\leq w(max\{f(x,T_1x), f(x, T_1x), f(T_1x, T_2T_1x), f(T_1x, T_1x), f(T_2T_1x, T_2T_1x)\})$.

$\leq w\ (max\ \{f(x,T_1x),\ f(x,\ T_1x),\ f(x,\ T_1x),\ f(T_1x,\ T_1x)\ ,\ f(T_2T_1x,\ T_2T_1x)\})$

By condition (1.2), we have

$$max\ \{f(x,T_1y),\ f(T_1x,T_1x),\ f(T_2T_1x,T_2T_1x)\} = f(x,T_1x).$$

then $f(x,T_1x) = f(T_1x,T_2T_1x) \leq w(f(x,T_1x)) < f(x,T_1x)$.

This is a contradiction. There for $x\ T_1x$.

Similarly, by taking $\{x_{(2p+1)2n+1}\}$ be the subsequence of $\{x_{(2p+1)2n}\}$,

We can show that $x = T_2x$, therefore x is a fixed point of T_1 and T_2, To show uniqueness, let $y \neq x$ be another fixed point of T_1 and T_2 then the condition (2.1) yields.

$f(x,y) = f(T_1x,T_2y)$

$\leq w\ (max\ \{f(x,y),\ f(x,\ T_1x),\ f(y,\ T_2y),\ f\ (y,\ T_1x)\ ,\ f(T_2y,\ T_2T_1x)\})$

$= w\ (max\ \{f(x,y),\ f(x,x),\ f(y,y),\ f(y,\ x),\ ,\ f(y,\ x)\})$, Rest part of the proof follows from the proof of the theorem 1.

Theorem 3. Let $T_1, T_2, T_3, \ldots, T_k$ be continuous self mappings on Hausdorff space x and f be a continuous, symmetric mapping of $X \times X$ into R^+ such that the conditions (1.1) , (1.2) hold and

(3.1) $f\ (T_ix,\ T_{i+1}y) \leq \omega(max\ \{f\ (x,y)\ ,\ f(x,T_ix)\ ,\ f(y,T_{i+1}y)\ ,\ f(y,T_ix)\ ,$
$f(T_{i+1}y, T_{i+1}T_ix)\})$

for all $x \neq y \in X$ and $T_{k+1} = T_1$..

If for some $x_0 \in X$, the sequence $\{x_n\}$, where

$$x_1 = T_1x_0,\ x_2 = T_2x_1\ \ldots,\ x_k = T_kx_{k-1}.$$
$$x_{k+1} = T_1x_k,\ x_{k+2} = T_2x_{k+1},\ \ldots,\ x_{2k} = T_k\,x_{2k-1}.$$
$$\ldots \quad \ldots \quad \ldots$$
$$x_{nk+1} = T_1x_{nk},\ x_{nk+2} = T_2x_{nk+1},\ \ldots,\ x_{(n+1)k} = T_k\,x_{(n+1)\,k}$$

for $n = 0, 1, 2, 3, 4$ .... has convergent subsequence $\{x_{(mk+1)\,n}\}$ where $m \in N$ is fixed and $n \in N$, then $T_1, T_2, T_3, \ldots, T_k$ have a unique common fixed point.

Proof. By applying (3.1), we get

$f\ (x_1,x_2) = f(T_1x_0, T_2x_1)$

$\leq w\ (max\ \{f(x_0,x_1),\ f(x_0,T_1x_0),\ f(x_1,\ T_2x_1),\ f\ (x_1,\ T_1x_0)\ ,\ f(T_2x_1,T_2\ T_1x_0)\})$

$= w(max\ \{f(x_0,x_1),\ f(x_0,x_1),\ f(x_1,\ x_2),\ f\ (x_1,\ x_1)\ f\ (x_2,\ x_2)\})$

which implies (as in the proof of theorem1)

$$f(x_1,x_2) \leq f(x_0,x_1)\ .$$

Similarly, we have

$$f(x_2,x_3) \leq f(x_1,x_2)$$

... ...

$$f(x_{k-1},x_{k}) \leq f(x_{k-2},x_{k-1}) .$$

Again, by applying (3.1), we have

$$f(x_k , x_{k+1}) = f(T_k x_{k-1}, T_1 x_k)$$
$$\leq w (max \{f(x_{k-1},x_k), f(x_{k-1}, T_k x_{k-1}), f(x_k, T_1 x_k), f(x_k ,T_k x_{k-1}) ,$$
$$f(T_1 x_k, T_1 T_k x_{k-1})\})$$
$$= w(max \{f(x_{k-1},x_k), f(x_{k-1}, x_k), f(x_k, x_{k+1}), f(x_k, x_k) f(x_{k+1}, x_{k+1})\}) .$$

This implies that

$$f(x_k,x_{k+1}) \leq f(x_{k-1},x_k)$$

and hence

$$f(x_n ,x_{n+1}) \leq f(x_{n-1} , x_n), n = 0, 1, 2, ...$$

By proceeding as in theorem 2, we can show that $T_1 ,T_2 ,T_3 ,...,T_k$ have a unique fixed point.

Acknowledgement

The authors would like to thank Prof. Bijendra Singh, School of Studies in Mathematics, Vikram University, Ujjain, M.P. for ncouragement in the preparation of this paper.

REFERENCES

[1] M. Chatterji and S.K. Ghoshal, Some fixed point theorem in Hausdorff spaces and consequences, *Indian J. Math.*, **22** (2) (1980), 2 – 6.

[2] Renu Chugh and Dinesh Rani, Some results on fixed points in a Hausdorff space, *Bull. Cal. Math.* **84** (1992), 219 – 226.

[3] B. Fisher and M.S. Khan, Pairwise contractive mappings on Hausdorff spaces, *Bull. Math. dela. Soc. Sci. Math dela R.S.* **25** (73) (1981), 37-40.

[4] V.Popa, Some unique fixed point theorems in Hausdorff spaces *Indian J. Pure and Appl. Math.* **14** (16) (1983), 713-717.

[5] Kiventidis Thomas, On fixed points in Hausdorff space, *Indian J. Pure and App. Math.* **16** (12) (1985), 1420-1424.

[6] J.S.W. Wong, Mapping of contractive type on abstract spaces *J. Anal. Appl.* (1972), 331-340.