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# DETERMINATION OF PHASE SHIFT DIFFERENCE FOR BINOMIAL POTENTIAL FUNCTION

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## ABSTRACT

In the present paper, we determine the phase shift diffrence for binomial potential function by making an appeal to technic of Tietz (1963).

**1. Introduction.** The general theory of scattering and experimental studies in the nuclear and atomic collision have great deal with development in the modern science and technology. Many of these theoretical and experimental advances have been the result of their mutual stipulation. The problem of deducing a potential from the obseved phase shifts has led to many mathematical investigations. However, it may be of interest to obtain phase shifts from a given potential function. Recently, a number of research scholars [3,5] have utilized Tietz [6] method for finding the phase shifts. The theoretical value of cross-section is determined with the help of phase shifts $n_L$, where '$L$' is the angular momentum quantum number. Bhattacharjie and Sudarshan [1] have evaluated a solvable potential function in the form

$$U(r) = B\frac{e^{\alpha r}}{1-e^{\alpha r}}, \qquad \text{... (1.1)}$$

for the $s$-wave Schrodinger equation by transforming second order Gauss equation, where $B$ and $\alpha$ are parameters.

Evidently, from (1.1), we get

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$$U(r) = \beta \sum_{n=1}^{\infty} e^{n\alpha r} \quad . \qquad \text{... (1.2)}$$

Therefore, from (1.2), we have

$$\frac{dU}{dr} = \sum_{n=1}^{\infty} A_n e^{n\alpha r} \quad . \qquad \text{... (1.3)}$$

Motivated by this work, here in the present paper, we may take a potential function in the following binomial form

$$U(r) = \lambda \sum_{n=1}^{\infty} (1-n\alpha r)^{-\beta} \quad , \qquad \text{... (1.4)}$$

where, $\alpha$ and $\beta$ are real or complex numbers and $\lambda$ is any constant. Also, from (1.4), we find

$$\frac{dU}{dr} = \sum_{n=1}^{\infty} \lambda_n (1-n\alpha r)^{-\beta-1} \quad . \qquad \text{... (1.5)}$$

where $\lambda_n$ is independent of $r$.

Obvisously, replacing $\alpha$ by $\alpha/\beta$ and $\beta \to \infty$, (1.4) gives (1.2) and (1.5) reduces to (1.3).

Here, in the present paper, we determine the phase shift difference for binomial potential function (1.4) by making an appeal to Tietz [6] technic.

**2. Formulae Used.** In our investigations, we shall need of the applications of following formulae :

According to Tietz's technique [6], for determining the phase shifts for the radial Schrodinger equation

$$\frac{d^2 U_L(r)}{dr^2} + \left[ K^2 - U(r) - \frac{L(L+1)}{r^2} \right] U_L(r) = 0 \, , \qquad \text{... (2.1)}$$

in order to obtain approximate formulae for smaller values of $L$, he assumed the phase shift difference $n_L - n_{L+1}$ to be small and proved the following useful integrals [6, p.291 (14) and p.292 (16)]

$$n_L - n_{L+1} = \frac{\pi}{2K} \int_0^{\infty} r \frac{dU}{dr} J_{L+1/2}(Kr) \, J_{L+3/2}(Kr) dr \quad , \qquad \text{... (2.2)}$$

and

$$n_{L-1} - n_{L+1} = \frac{(L+1/2)\pi}{K^2} \int_0^{\infty} \frac{dU}{dr} J^2_{L+1/2}(Kr) \, dr \quad , \qquad \text{... (2.3)}$$

where $J_v(Kr)$ is the Bessel function of the first kind, $K$ and $U(r)$ denote the terms for the total energy and potential energy respectively.

Also Due to Luke [2, p. 24(17)], we write

$$J_\nu (az)\ J_\mu (bz) = \frac{(az/2)^\nu (Bz/2)^\mu}{\Gamma(\nu+1)\,\Gamma(\mu+1)} \sum_{n=0}^{\infty} \frac{(-1)^n (az/2)^{2n}}{n!\,(\nu+1)_n}\ {}_2F_1\left[\begin{matrix} -n, -\nu-n\ ; \\ \mu+1; \end{matrix}\ \frac{b^2}{a^2}\right],$$

provided that $Re\ (\nu+\mu+1) > 0.$ ... (2.4)

Now, setting $a=b=K$ and replacing $z$ by $r$, then making an appeal to the formula due to Rainville [4, p. 49] and (2.4), we derive

$$J_\nu (Kr)\ .\ J_\mu (Kr) = \frac{(Kr/2)^{\nu+\mu}}{\Gamma(\nu+1)\,\Gamma(\mu+1)}\ {}_2F_3\left[\begin{matrix} (\nu+\mu+1)/2\ ,\ (\nu+\mu+2)/2\ ; \\ \nu+1,\ \mu+1,\ \nu+\mu+1; \end{matrix}\ -K^2r^2\right]$$

where $Re\ (\nu+\mu+1) > 0.$ ... (2.5)

Now, on substituting $\nu = L + {}^1/_2$ and $\mu = L + {}^3/_2$ in (2.5), then using the results (2.2) and (2.5), we obtain

$$n_L - n_{L+1} = \frac{\pi K^{2L+1}}{2^{2L+3}\ \Gamma(L+3/2)\,\Gamma(L+5/2)}$$

$$.\int_0^\infty \frac{dU}{dr}\, r^{2L+3}\ {}_1F_2\left[\begin{matrix} L+2\ ; \\ L+5/2,\ 2L+3\ ; \end{matrix}\ -K^2r^2\right] dr$$

provided that $Re\ (2L+3) > 0$ . ... (2.6)

Further, choosing $\nu = \mu = L+1/2.$ in (2.5), then making an appeal to (2.3) and (2.5), we evaluate

$$n_{L-1} - n_{L+1} = \frac{\pi K^{2L+1}}{2^{2L+1}\ \Gamma(L+1/2)\,\Gamma(L+3/2)}$$

$$.\int_0^\infty \frac{dU}{dr}\, r^{2L+1}\ {}_1F_2\left[\begin{matrix} L+1\ ; \\ L+3/2,\ 2L+2\ ; \end{matrix}\ -K^2r^2\right] dr\ ,$$

provided that $Re\ (2L+2) > 0$ . ... (2.7)

**3. Determination of Phase-Shift Difference for Binomial Potential Function.** In this section, we determine the phase shifts for binomial potential function (1.4), Using Tietz's technique [6].

In order to obtain the phase shifts, we make an appeal to (1.5) and (2.6) and derive

$$n_L - n_{L+1} = \frac{\pi K^{2L+1}}{2^{2L+3}\ \Gamma(L+3/2)\,\Gamma(L+5/2)}$$

$$\cdot \sum_{n'=1}^{\infty} \lambda_{n'} \int_0^{\infty} (1-n\,\alpha r)^{-(\beta+1)} r^{2L+3} \, {}_1F_2\left[\begin{matrix} L+2\,; \\ L+5/2,\ 2L+3\,; \end{matrix} -K^2r^2\right] dr$$

or

$$n_L - n_{L+1} = \frac{\pi K^{2L+1}}{2^{2L+3}\ \Gamma(L+3/2)\,\Gamma\,(L+5/2)} \sum_{n'=1}^{\infty} \sum_{m=0}^{\infty} \lambda_n \frac{(L+2)_m\,(-K^2)^m}{(L+5/2)_m\ (2L+3)_m\ m!}$$

$$\cdot \int_0^{\infty} \frac{r^{2L+2m+3}}{(1-n\alpha r)^{\beta+1}}\ dr\ . \qquad \ldots (3.1)$$

Now, substituting $n\alpha r = -y$ from (3.1), we have

$$n_L - n_{L+1} = \frac{\pi K^{2L+1}}{2^{2L+3}\ \Gamma(L+3/2)\,\Gamma\,(L+5/2)} \sum_{n'=1}^{\infty} \sum_{m=0}^{\infty}$$

$$\frac{\lambda_n\,(L+2)_m\,(-K^2/n^2\alpha^2)^m}{(-n\alpha)^{2L+4}(L+5/2)_m\ (2L+3)_m\ m!} \int_0^{\infty} \frac{y^{(2L+2m+4)\,-1}}{(1+y)^{(2L+2m+4)\,+(\beta-2L-2m-3)}}\,dy \ \ldots (3.2)$$

Now, making an appeal to the well known result

$$\int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}}\,dy = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma\,(m+n)}.$$

provided that $m > 0$ and $n > 0$ , ... (3.3)

(3.2) finally gives

$$n_L - n_{L+1} = \frac{\sqrt{\pi}\ K^{2L+1}\ \ \Gamma(L+2)\ \Gamma\ (\beta-2L\ -3)}{\Gamma(L+3/2)\,\Gamma\,(\beta+1)}$$

$$\cdot \sum_{n=1}^{\infty} \frac{\lambda_n}{(-n\alpha)^{2L+4}}\ {}_2F_3\left[\begin{matrix}(L+2)\,,\,(L+2)\,; \\ (2L+3),(L+2-\beta/2),(L+5/2-\beta/2)\,;\end{matrix} -\frac{K^2}{n^2\alpha^2}\right] dr$$

provided that $(2L+3) > 0$, $(2L+4-\beta) > 0$ and $0 < \beta < 1$ and Re $(n'\alpha \pm iK) > 0$. ... (3.4)

Further, Making an appeal to (1.5) and (2.7), we obtain

$$n_{L-1} - n_{L+1} = \frac{\pi K^{2L-1}}{2^{2L+1}\ \Gamma(L+1/2)\,\Gamma\,(L+3/2)}$$

$$\cdot \sum_{n=1}^{\infty} \lambda_n \int_0^{\infty} r^{2L+1}\,(1-n\alpha r)^{-\beta-1}\ {}_1F_2\left[\begin{matrix}L+1\,; \\ (L+3/2),\,(2L+2)\,;\end{matrix} -K^2r^2\right] dr. \quad \ldots (3.5)$$

Now, replacing $-n\alpha r$ by $y$ and using the results (3.3) and (3.5), then after some specializatons, we evaluate

$$n_{L-1} - n_{L+1} = \frac{\sqrt{\pi}\ K^{2L-1}\ \ \Gamma(L+1)\ \Gamma\ (\beta-2L\ -1)}{\Gamma(L+1/2)\,\Gamma\,(\beta+1)}$$

$$\cdot \sum_{n=1}^{\infty} \frac{\lambda_n}{(-n)^{2L+2}} \; {}_2F_3 \left[\begin{matrix}(L+1), (L+1); & \\ (2L+2), (L+1-\beta/2), (L+3/2-\beta/2); & -\frac{K^2}{n^2\alpha^2}\end{matrix}\right]$$

provided that $Re\ (2L+2) > 0$, $\text{Re}(2L+2-\beta) > 0$ and $0 < \beta < 1$ and $Re(n'\alpha \pm iK) > 0$. ...(3.6)

The formulae (3.4) and (3.6) establish an already known fact that for low energies and larger values of $L$, the phase shifts difference is an odd function of $K$. Futher for physical interest the phase shifts are to be finite.

**4. Special Cases.**

For $L = 0$, we get from (3.4)

$$n_0 - n_1 = \frac{\sqrt{\pi}\, K\Gamma(2)\, \Gamma\,(\beta-3)}{\Gamma(3/2)\, \Gamma\,(\beta+1)} \sum_{n=1}^{\infty} \frac{\lambda_n}{(-n\alpha)^4} \cdot {}_2F_3\left[\begin{matrix}2, 2; & \\ 3, (\frac{4-\beta}{2}), (\frac{5-\beta}{2}) & -\frac{K^2}{n^2\alpha^2}\end{matrix}\right]$$

or

$$n_0 - n_1 = \frac{4K\Gamma(2)\, \Gamma\,(\beta-3)}{\alpha^4\, \Gamma\,(\beta+1)} \sum_{n=1}^{\infty} \lambda_n \left[\frac{1}{2n^4} + \frac{2\,(-\frac{K^2}{n^2\alpha^2})}{3n^4(\frac{4-\beta}{2})\,(\frac{5-\beta}{2})} \right.$$

$$\left. + \frac{3\,(-\frac{K^2}{n^2\alpha^2})^2}{3n^4(\frac{4-\beta}{2})\,(\frac{5-\beta}{2})\,(\frac{6-\beta}{2})\,(\frac{7-\beta}{2})} + \ldots \right] \qquad \ldots (4.1)$$

Further, taking $L = 1$ in (3.4), we have

$$n_1 - n_2 = \frac{\sqrt{\pi}\, K^3\,\Gamma(3)\, \Gamma\,(\beta-5)}{\Gamma(^3/_2+1)\, \Gamma\,(\beta+1)} \sum_{n=1}^{\infty} \frac{\lambda_n}{(-n\alpha)^6} \cdot {}_2F_3\left[\begin{matrix}3, 3; & \\ 5, (\frac{6-\beta}{2}), (\frac{7-\beta}{2}); & -\frac{K^2}{n^2\alpha^2}\end{matrix}\right].$$

Therefore,

$$n_1 - n_2 = \frac{4K\Gamma(2)\, \Gamma\,(\beta-3)}{\alpha^4\, \Gamma\,(\beta+1)} \sum_{n=1}^{\infty} \lambda_n \left[ - \frac{1\,(-\frac{K^2}{n^2\alpha^2})}{6n^4(\frac{4-\beta}{2})\,(\frac{5-\beta}{2})} \right.$$

$$\left. - \frac{3\,(-\frac{K^2}{n^2\alpha^2})^2}{10\, n^4(\frac{4-\beta}{2})\,(\frac{5-\beta}{2})\,(\frac{6-\beta}{2})\,(\frac{7-\beta}{2})} + \ldots \right]. \qquad \ldots (4.2)$$

Now, adding (4.1) and (4.2), we find

$$n_0 - n_2 = \frac{4K\Gamma(2)\, \Gamma\,(\beta-3)}{\alpha^4\, \Gamma\,(\beta+1)} \sum_{n=1}^{\infty} \lambda_n \left[\frac{1}{2n^4} + \frac{1\,(-\frac{K^2}{n^2\alpha^2})}{2n^4(\frac{4-\beta}{2})\,(\frac{5-\beta}{2})} \right.$$

$$+\frac{9\left(-\frac{K^2}{n^2\alpha^2}\right)^2}{20\,n^4\left(\frac{4-\beta}{2}\right)\left(\frac{5-\beta}{2}\right)\left(\frac{6-\beta}{2}\right)\left(\frac{7-\beta}{2}\right)}+\dots\Bigg]. \qquad \dots (4.3)$$

Now in (3.6), setting $L=1$, we derive

$$n_0 - n_2 = \frac{\sqrt{\pi}\,K\Gamma(2)\,\Gamma(\beta-3)}{\Gamma(3/2)\,\Gamma(\beta+1)} \sum_{n=1}^{\infty} \frac{\lambda_n}{(-n\alpha)^4}\ {}_2F_3\left[\begin{matrix}2\,,\,2\,; \\ 4,\,(\frac{4-\beta}{2}),\,(\frac{5-\beta}{2})\,;\end{matrix} -\frac{k^2}{n^2\alpha^2}\right]$$

which also further gives (4.3). ... (4.4)

If $n_0$ is known, all the other values of phase shifts can easily be obtained from the relations (3.4) and (3.6).

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