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MEIJER'S *G*-FUNCTION OF ONE VARIABLE AND ITS APPLICATIONS IN TWO BOUNDARY VALUE PROBLEMS

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ABSTRACT

The aim of this paper is to solve two boundary value problems on (i) heat conduction in a rod (ii) deflection of vibrating string, employing Meijer's *G*-function of one variable.

1. Introduction. The Meijer's *G*-function of one variable is given in Erdélyi [2. p. 207], defined as follows :

$$G^{m,n}_{p,q}[x] = G^{m,n}_{p,q}\left[x \middle| \begin{matrix} a_p \\ b_q \end{matrix}\right] \qquad \text{... (1.1)}$$

$$= \frac{1}{2\pi i} \int_L \theta(\xi)\, x^{\xi}\, d\xi ,$$

where
$$\theta(\xi) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \xi) \prod_{j=1}^{n} \Gamma(1 - a_j + \xi)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + \xi) \prod_{j=n+1}^{p} \Gamma(a_j - \xi)} .$$

and $i = \sqrt{(-1)}$

2. Formula Required. In this paper, we shall make application of the following integral [1,p. 123, (2.1)] :

$$\int_0^L \left(\sin \frac{\pi x}{L}\right)^{w-1} \sin \frac{\lambda_m \pi x}{L}\, G^{u,v}_{p,q}\left[z\left(\sin \frac{\pi x}{L}\right)^{2d} \middle| \begin{matrix} a_p \\ b_q \end{matrix}\right] dx$$

$$= \frac{L \sin \lambda_m \pi/2.}{\sqrt{(d\pi)}}\, G^{u,\, v+2d}_{p+2d,\, q+2d}\left[z \middle| \begin{matrix} \Delta\,(2d,\, 1\text{-}w),\, ap \\ bq,\, \Delta\,(d,\, \frac{1-w-\lambda m}{2}),\, \Delta\,(d\,,\, \frac{1-w+\lambda m}{2}) \end{matrix}\right] \qquad \text{...(2.1)}$$

Problem -1

3. Application to heat conduction in a Rod. In this section, we consider a problem on outer heat conduction in a rod under certain boundary conditions. If the thermal coefficients are constant and there is no source of thermal energy, then the temperature $u\ (x,t)$ in a one dimensional rod $0 \leq x \leq L$ satisfies the following heat equation :

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2},\ t \geq 0 \qquad \text{... (3.1)}$$

If we take the following boundary conditions

$$u\ (0\ ,t) = 0, \qquad \text{... (3.2)}$$

$$\frac{\partial u\ (L,t)}{\partial x} + hu(L,t) = 0, \qquad \text{... (3.3)}$$

$u\ (x,t)$ is finite as $t \to \infty$,

and initial condition

$$u\ (x,0) = f(x), \qquad \text{... (3.4)}$$

then the solution of partial differential equation (3.1) is given by [3, p. 77 (4)]

$$u\ (x,t) = \sum_{m=1}^{\infty} A_m \sin \frac{\lambda_m \pi x}{L} \exp\left\{-\left(\frac{\pi\lambda_m}{L}\right)^2 kt\right\}, \qquad \text{... (3.5)}$$

where λ_1, λ_2 , ... , λ_m are the roots of the transcendental equation

$$\tan \pi\lambda_m = \frac{\pi\lambda_m}{kL}. \qquad \text{... (3.6)}$$

Now we shall consider the problem of determining $u\ (x,t)$, where

$$u\ (x,0) = f(x) = \left(\sin \frac{\pi x}{L}\right)^{w-1} G^{u,\ v}_{p,\ q}\left[z\left(\sin \frac{\pi x}{L}\right)^{2d} \middle| \begin{matrix} a_p \\ b_q \end{matrix}\right]. \qquad \text{... (3.7)}$$

4. Solution of the Problem. Combining (3.5) and (3.7) and making the use of integral (2.1), we drive

$$A_m = \frac{4\pi \sin \lambda_m \pi/2}{\sqrt{(d\pi)}\ [2\pi\lambda_m - \sin 2\pi\lambda_m]}.$$

$$G^{u,\ v+2d}_{p+2d,\ q+2d}\left[z \middle| \begin{matrix} \Delta\ (2d,\ 1-w),\ a_p \\ b_q, \Delta\ (d,\ \frac{1-w-\lambda m}{2}),\ \Delta\ (d\ ,\ \frac{1-w+\lambda m}{2}) \end{matrix}\right]. \qquad (4.1)$$

Putting the value of A_m from (4.1) in (3.5), we get the following required solution of the problem :

$$u\ (x,t) = \frac{4\pi}{\sqrt{(d\pi)}} \sum_{m=1}^{\infty} \frac{\lambda_m \sin \frac{\pi\lambda_m}{2} \sin \frac{\lambda_m \pi x}{L}}{[2\pi\lambda_m - \sin 2\pi\lambda_m]} \cdot \exp\left\{-\left(\frac{\pi\lambda_m}{L}\right)^2 kt\right\}$$

$$G^{u,\, v+2d}_{p+2d,\, q+2d}\left[z \,\middle|\, \begin{matrix} \Delta\,(2d,\, 1-w),\, a_p \\ b_q,\, \Delta\,(d,\, \frac{1-w-\lambda m}{2}),\, \Delta\,(d\,,\, \frac{1-w+\lambda m}{2}) \end{matrix} \right]. \tag{4.2}$$

5. Expansion Formula. Making an use of (3.7) and (4.1) in (3.5) we derive the following expansion formula

$$(sin\,\frac{\pi x}{L})^{w-1}\; G^{u,\, v}_{p,\, q}\left[z(sin\,\frac{\pi x}{L})^{2d} \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix} \right]$$

$$= \frac{4\pi}{\sqrt{(d\pi)}} \sum_{m=1}^{\infty} \frac{\lambda_m sin\,\frac{\pi\lambda_m}{2}\; sin\,\frac{\lambda_m \pi x}{L}}{[2\pi\lambda_m - sin 2\pi\lambda_m]}\,.$$

$$G^{u,\, v+2d}_{p+2d,\, q+2d}\left[z \,\middle|\, \begin{matrix} \Delta\,(2d,\, 1\text{-}w),\, a_p \\ b_q,\Delta\,(d,\, \frac{1-w-\lambda m}{2}),\, \Delta\,(d\,,\, \frac{1-w+\lambda m}{2}) \end{matrix} \right]. \qquad ...(5.1)$$

Problem -2

6 Application to Homogeneous Wave Problem. In this section, we shall determine the deflection $u\,(x,t)$ of vibrating string. If the deflection due to the weight of string is negligible, then $u\,(x,t)$ satisfies the partial differential equation.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c}\,\frac{\partial^2 u}{\partial t^2},\; 0 < x < L,\; t > 0. \qquad ...\,(6.1)$$

Now we assume the boundary conditions

$$u\,(0\,,t) = 0,\; u\,(L,t) = 0,\; t > 0, \qquad ...\,(6.2)$$

and initial conditions

$$\frac{\partial u}{\partial t}\,(x,0) = g(x)\;(\text{initial velocity}) \qquad ...\,(6.3)$$

and

$$u\,(x,0) = f(x). \qquad ...\,(6.4)$$

Then the solution of partial differential equation (6.1) is given by

$$u\,(x,t) = \sum_{m=1}^{\infty} [a_m\, cos\,\frac{\pi\lambda_m ct}{L} + bm\; sin\frac{\pi\lambda_m ct}{L}]\; sin\,\frac{\pi\lambda_m x}{L}\,. \qquad ...\,(6.5)$$

Now we consider the problem of determining $u\,(x,t)$, where $u\,(x,0) = f(x)$ is given by (3.7) while

$$g\,(x) = (sin\,\frac{\pi x}{L})^{w-1}\; G^{M,\, N}_{P,\, Q}\left[z(sin\,\frac{\pi x}{L})^{2\xi} \,\middle|\, \begin{matrix} A_p \\ B_q \end{matrix} \right] \qquad ...(6.6)$$

By (6.3) , (6.4) and (6.5), it is clear that

$$u\,(x,0) = f(x) = \sum_{m=1}^{\infty} a_m\, sin\,\frac{\pi\lambda_m x}{L}\,. \qquad ...\,(6.7)$$

and

$$\frac{\partial u}{\partial x}(x,0) = g(x) = \frac{\pi c}{L} \sum_{m=1}^{\infty} b_m \lambda_m \sin \frac{\pi \lambda_m x}{L} \quad . \qquad \ldots (6.8)$$

Now making use of the integral (2.1), we find the values of a_m and b_m separately and put them in (6.5) to get required solution of the problem in the following form :

$$u(x,t) = \frac{4\pi}{\sqrt{(d\pi)}} \sum_{m=1}^{\infty} \frac{\lambda_m \sin \frac{\pi\lambda_m}{2} \cos \frac{\pi\lambda_m ct}{L} \sin \frac{\lambda_m \pi x}{L}}{[2\pi\lambda_m - \sin 2\pi\lambda_m]} .$$

$$G^{u,\, v+2d}_{p+2d,\, q+2d}\left[z \,\middle|\, \begin{matrix} \Delta(2d, 1-w), a_p \\ b_q, \Delta\left(d, \frac{1-w-\lambda m}{2}\right), \Delta\left(d, \frac{1-w+\lambda m}{2}\right) \end{matrix} \right]$$

$$+ \frac{4L}{c\sqrt{(d\pi)}} \sum_{m=1}^{\infty} \frac{\lambda_m \sin \frac{\pi\lambda_m}{2} \sin \frac{\pi\lambda_m ct}{L} \sin \frac{\lambda_m \pi x}{L}}{[2\pi\lambda_m - \sin 2\pi\lambda_m]} .$$

$$G^{M,\, N+2\xi}_{P+2\xi,\, Q+2\xi}\left[z \,\middle|\, \begin{matrix} \Delta(2\xi, 1-w'), A_p \\ B_q, \Delta\left(\xi, \frac{1-w'-\lambda m}{2}\right), \Delta\left(\xi, \frac{1-w'+\lambda m}{2}\right) \end{matrix} \right] \qquad (6.9)$$

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