Jñānābha, Vol. 28, 1998

# SOLVABLE POTENTIAL FOR THE WEBER - HERMITE EQUATION

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*(Received : June 25, 1997)*

**ABSTRACT**

The aim of this paper is to evaluate solvable potential for the Weber -Hermite equation.

**1. Introduction.** Recently, Bhattacharijie and Sudarshan [1] formulated a general method for evaluating solvable potentials for the $S$-wave Schrödinger equation. Following them in this paper we evaluate solvable potential for the $S$-wave Schrödinger equation by making use of second order Weber - Hermite diffrential equation ; we also give out line the derivation of this potential by the alternative method developed by Bose. [2].

**2. Potential for the Weber -Hermite Equation .** The function $D_\upsilon(z)$ and $D_{-\upsilon-1}(iz)$ are two linearly independent solutions of Weber-Hermite equation [4.p.116-117]

$$y''(z) + (\upsilon + 1/2 - 1/4\, z^2)y = 0 \qquad \text{... (2.1)}$$

for all value of $\upsilon$,

where $\quad D_\upsilon(z) = 2^{(\upsilon-1)/2}\, e^{-z^2/4}\, z\, \psi\, (1-\upsilon)/2\, ;\, 3/2;\, z^2/2).$

The general solution of (2.1) is given by

$$\begin{aligned} y(z) &= c_1\, D_\upsilon(z) + c_2\, D_{-\upsilon-1}(iz), \qquad \text{...(2.2)} \\ &= c_1\, 2^{(\upsilon-1)/2}\, e^{-z^2/4}\, z\, \psi\, (1-\upsilon)/2\, ;\, 3/2;\, z^2/2) \\ &\quad + c_2\, 2^{(-\upsilon-2)/2}\, e^{-z^2/4}\, (iz)\, \psi\, (1+\upsilon/2\, ;\, 3/2;\, -z^2/2), \end{aligned}$$

where $c_1$ and $c_2$ are arbitrary constans.

Following Bhattacharjie and Sudarshan [1, p. 864], (2.1) can be transformed to followeing $S$-wave radial Schrödinger equation (adopting units such that $E = K^2$) ;

$$\phi''(r) + [k^2 - V(r)]\,\phi(r) = 0. \qquad \text{... (2.3)}$$

By making the following substitutions

$$z = f(r),\ y(z) = g(r)\,\phi(r),\ h(r) = \frac{d\,\{\log g(r)\}}{dr} \qquad \text{... (2.4)}$$

(2.1) takes the form

$$\phi''(r) + A(r)\,\phi(r) + B(r)\,\phi(r) = 0, \qquad \text{... (2.5)}$$

where

$$A(r) = 2\frac{g'(r)}{g(r)} - \frac{f''(r)}{f'(r)}, \qquad \text{...(2.6)}$$

$$B(r) = \frac{g''(r)}{g(r)} - \frac{f''(r)}{f'(r)} \cdot \frac{g'(r)}{g(r)} + [\upsilon + 1/2 - \frac{\{f(r)\}^2}{4}]\,\{f'(r)\}^2 \qquad \text{...(2.7)}$$

Now for (2.5) to be of form (2.3), the following conditions should be satisfied :

$$A(r) = 0,\ B(r) = k^2 - V(r),\ \frac{\partial\{V(r)\}}{\partial k} = 0. \qquad \text{...(2.8)}$$

The third condition of (2.8) ensures the independence of $v(r)$ from $k$ completely.

Substituting the first condition of (2.8) into (2.6) and integrating, we get

$$f'(r) = Mg^2(r), \qquad \text{... (2.9)}$$

where $M$ is the constant of integration.

Similarly, from the second condition of (2.8), (2.7) and (2.4), we arrive at

$$h'(r) - h^2(r) + [\upsilon + 1/2 - \frac{\{f(r)\}^2}{4}]\,\{f'(r)\}^2 \equiv k^2 - V(r) \qquad \text{... (2.10)}$$

Considering as a particular choice

$$z = f(r) = 1 - e^{-ar}, \qquad \text{... (2.11)}$$

we obtain from (2.10),

$$a^2 e^{-2ar}\,[(\upsilon + 1/4) - \frac{e^{-ar}}{2}(\frac{e^{-ar}}{2} - 1)] - \frac{a^2}{4} \equiv k^2 - V(r), \qquad \text{...(2.12)}$$

which suggests

$$k^2 = -\frac{a^2}{4}, \qquad \text{... (2.13)}$$

and

$$V(r) = a^2 e^{-2ar}\,[-(\upsilon + 1/4) + \frac{e^{-ar}}{2}(\frac{e^{-ar}}{2} - 1)] \qquad \text{... (2.14)}$$

Now for the potential derived in (2.14), using (2.2) the general solution $\phi(r)$ of the Schrödinger eqution (2.3) can be written as

$$\phi(r) = c_1 e^{ikr} \exp\{-1/4\,(1-e^{-2ikr})^2\}\{1-e^{-2ikr}\}\ \psi\left((1-\upsilon)/2\,;\,3/2\,;\,\frac{(1-e^{-2ikr})^2}{2}\right)$$

$$+ c_2 e^{-ikr} \exp\{(1-e^{2ikr})^2/4\}\{i(1-e^{-2ikr})\}\ \psi\left((1+\upsilon)/2;\,3/2\,;\,\frac{-(1-e^{2ikr})^2}{2}\right) \quad (2.15)$$

where the constant $\sqrt{M}$ being included in new constants $c_1$ and $c_2$. Constants $c_1$ and $c_2$ are to be determined, so that $\phi(0) = 0$ and $\phi(r)$ is continual normalized.

Now $(1-e^{\pm 2ikr}) = 1 - \cos 2kr \mp i \sin 2kr$

Let $1-\cos 2kr \sim \beta\ (r \to \infty)$

and $\sin 2kr \sim \delta\ (r \to \infty)$

then, asymptotically,

$$\phi(r) \sim N_1 e^{ikr}\, \psi\left((1-\upsilon)/2\,;\,3/2\,;\,\frac{(\beta+i\delta)^2}{2}\right)$$

$$+ N_2 e^{-ikr}\, \psi\left(1+\upsilon/2\,;\,3/2\,;\,\frac{-(\beta-i\delta)^2}{2}\right),\quad r \to \infty \qquad \text{... (2.16)}$$

where

$$N_1 = c_1 \left[\exp\left\{-\frac{(\beta+i\delta)^2}{4}\right\}\right](\beta+i\delta) \quad ,$$

$$N_2 = c_2 \left[\exp\left\{\frac{(\beta-i\delta)^2}{4}\right\}\right][i(\beta-i\delta)].$$

It is to be noted that value of $\beta$ oscillates between $0$ and $2$. When $r$ tends to $\infty$ and it is likely to become $0$ at some infinite points. Only such points enable us to evaluate the $S$-matrix.

Thus, when $\beta = 0$, (2.16) takes the form

$$\phi(r) \sim N_1 e^{ikr}\, \psi((1-\upsilon)/2\,;\,3/2\,;-\delta^2/2)$$

$$+ N_2 e^{-ikr}\, \psi(1+\upsilon/2\,;\,3/2\,;\,\delta^2/2),\quad r \to \infty. \qquad \text{... (2.17)}$$

(2.17) together with the requirement $\phi(r) = 0$ at $r=0$, yields the $S$-matrix.

$$S(k) = \frac{-N_2}{N_1} = \frac{\psi((1-\upsilon)/2\,;\,3/2\,;-\delta^2/2)}{\psi(1+\upsilon/2\,;\,3/2\,;\,\delta^2/2)}\ . \qquad \text{... (2.18)}$$

It should be noted, however, that the above expression for the $S$-matrix is valid only for such values of $\beta$ which tend to zero when $r \to \infty$.

**3. Potential by an Alternative Method.** The normal form of an ordinary differential equation in the notation of Bose [2, p. 245] is given by

$$V''(z) + I(z)\,V(z) = 0. \qquad \text{...(3.1)}$$

Putting $z = z(r)$ and $V(z) = [z'(r)]^{\frac{1}{2}} f(r)$, we get

$$f''(r) + I_S(r) f(r) = 0. \quad \ldots(3.2)$$

where $I_S(r) = [z'(r)]^2 I(z) + 1/2\{z,r\}$, ... (3.3)

and $\{z, r\}$ is the Schwarzian derivative due to Erdélyi [3].

Now $I_s(r)$ of the $S$-wave radial Schrödinger equation is given by

$$I_S(r) = k^2 - V(r). \quad \ldots(3.4)$$

Thus combining (3.3) and (3.4) we get

$$[z'(r)]^2 I(z) + 1/2\{z,r\} = k^2 - V(r). \quad \ldots(3.5)$$

The $I(z)$ of the Weber - Hermite equation (2.1) is given as

$$I(z) = \upsilon + 1/2 - z^2/4. \quad \ldots(3.6)$$

Therefore, again for the choice $z = f(r) = 1 - e^{-ar}$, using (3.3) and (3.6) we obtain

$$I_S(r) = -\frac{a^2}{4} + a^2 e^{-2ar}\left[(\upsilon + 1/4) - \frac{e^{-ar}}{2}\left(\frac{e^{-ar}}{2} - 1\right)\right]$$

and hence we finally arrive at the potential given in (2.14)

**4. Special Case.** If we let $z = f(r) = \beta r$ in (2.10) we shall obtain the potential for Weber - Hermite equation given as follows :

$$V(r) = \frac{\beta^2 r^2}{4}. \quad \ldots(4.1)$$

## REFERENCES


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