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ON KAEHLERIAN SPACES ADMITTING A ONE PARAMETER CONFORMAL TRANSFORMATION GROUP

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1. Introduction. Let K_n be a connected $(C^{\infty}-)$ differentiable Kaehlerian space of dimension n and g_{ji}, ∇_j, $R^h{}_{kji}$, R_{ji} and R, respectively the components of metric tensor field, the operator of covariant differentiation with respect to the Levi–Civita connection, the curvature tensor field, the Ricci tensor field and the scalar curvature field. Here and hereafter, the indices $a, b, c, \ldots, i, j, k, \ldots$ run over the range $1,2,3, \ldots, n$. We shall denote $g^{ja}\nabla_a$ by ∇^j and the Laplace – Beltrami operator by Δ. Throughout this paper, we assume that Kaehlerian spaces are connected and differentiable and functions are also differentiable.

An infinitesimal transformation v^h on K_n is said to be conformal, if it satisfies

(1.1) $£_v\, g_{ji} = \nabla_j v_i + \nabla_i v_j = 2\rho g_{ji}$,

for some function ρ on K_n, where $£_v$ denotes the operator of Lie–derivation with respect to v^h and $v_j = g_{ja} v^a$. The ρ satisfies $\rho = \nabla_a v^a / n$. If ρ in (1.1) is a constant, the transformation is said to be homothetic and if $\rho = 0$, the transofromation is called to be isometric. Hereafter, we shall denote the gradient of ρ by $\rho_j = \nabla_j \rho$.

We, now, put

$$G_{ji} = R_{ji} - Rg_{ji}/n, \quad Z_{Kjih} = R_{kjih} - R\,(g_{Kh}\, g_{ji} - g_{jh}\, g_{ki})/n\,(n-1).$$

We then have

(1.2) $G_{ji}\, g^{ji} = 0$, $\quad Z^a{}_{aji} - G_{ji}$.

In 1968, Yono and Sawaki [2] introduced the covariant tensor field

(1.3) $W_{kjih} = a\, Z_{kjih} + b\,(g_{Kh}\, G_{ji} - g_{jh}\, G_{ki} + G_{Kh}\, g_{ji} - G_{jh}\, g_{ki})/\,(n-2)$,

where a and b being constants, not both zero. It is easily seen that

$$W_{kjih}\, g^{kh} = (a+b)\, G_{ji} \ .$$

Hereafter, we shall use the following notations :

$$f = G_{ji}\, G^{ji}, \; z = Z_{kjih}\, Z^{kjih}, \; w = W_{kjih}\, W^{kjih} \ .$$

In 1968, Yano and Sawaki [2] proved the following theorems:

Theorem (1.1). Suppose that a compact orientable Riemannian space M_n with constant scalar curvature field R and of dimension >2 satisfies

$$\alpha_0 f + \beta_0 z - \alpha_1 \Delta f - \beta_1 \Delta z = constant,$$

where $\alpha_0, \alpha_1, \beta_0$ and β_1 are non–negative constants, not all zero, such that if $n > 6$.

(1.4) $8R(n-1)^{-1}\alpha_1 \geq (n-6)\alpha_0 \geq 0,$

$8R(n-1)^{-1}\beta_1 \geq (n-6)\beta_0 \geq 0.$

If M_n admits and infinitesimal non–isometric conformal transformation $v^h: \pounds_v g_{ji} = 2\rho g_{ji}, \ \rho \neq 0$, then M_n is isometric to a sphere.

Theorem (1.2): If a compact orientable Riemannian space M_n with constant curvature field R and of dimension >2 admits an infinitesimal non-isometric formal transformation $v^h: \pounds_v g_{ji} = 2\rho g_{ji}, \rho \neq 0$, such that

$\pounds_v \pounds_v (\alpha_0 f + \beta_0 z + \alpha_1 \Delta f + \beta_1 \Delta z) \leq 0,$

where $\alpha_0, \alpha_1, \beta_0$ and β_1 are non–negative constants, not all zero, such that

(1.5) $4(n-1)R^{-1}\alpha_0 \geq (n+6)\alpha_1 \geq 0,$

$4R(n-1)R^{-1}\beta_0 \geq (n+6)\beta_1 \geq 0.$

then M_n is isometric to a sphere.

Theorem (1.3). Suppose that a compact orientable Riemannian space M_n with constant scalar curvature field R and of dimension >2 admits an infinitesimal non–isometric conformal transformation

$v^h: \pounds_v g_{ji} = 2\rho g_{ji}, \rho \neq 0.$

If $\pounds_v \pounds_v \ w = 0$, a and b being constants such that $a + b \neq 0$, then M_n is isometric to a sphere.

The purpose of the present paper is to prove the following theorems in Kaehlerian spaces K_n admitting one–parameter conformal transformation of group

Theorem (A). If a compact orientable Kaehlerian space K_n with constant scalar curvature field R and of dimension >2 admits an infinitesimal non – isometric conformal transformation

$v^h: \pounds_v g_{ji} = 2\rho g_{ji}, \rho \neq 0$. then

(1.6) $\int_{Kn} \pounds_v \pounds_v (\alpha_0 f + \beta_0 z - \alpha_1 \Delta f - \beta_1 \Delta z)\, dv$

$$\geq \frac{n\,(n+2)}{2} \int_{Kn} \rho^2(\alpha_0 f + \beta_0 z - \alpha_1 \Delta f - \beta_1 \Delta z)\, dv$$

holds, where dv denotes the volume element of K_n and α_0 ,α_1,β_0 and β_1 are non–negative constants, not all zero, such that if $n > 6$, (1.4) holds, the equality in (1.6) holds if and only if K_n is isometric to a sphere.

This theorem is the generalization of theorem (1.1).

Theorem (B). If a compact orientable Kaehlerian space K_n with constant scalar curvature field R and of dimension > 2 admits an infinitesimal non–isometric conformal transformation
$v^h : \pounds_v g_{ji} = 2\rho g_{ji}$, $\rho \neq 0$, then

(1.7) $\int_{Kn} \pounds_v \pounds_v (\alpha_0 f + \beta_0 z + \alpha_1 \Delta f + \beta_1 \Delta z)\, dv \geq 0$,

holds, where α_0 ,α_1,β_0 and β_1 , are non–negative constants, not all zero, such that (1.5) holds, the equlity in (1.7) holds if and only if K_n is isometric to a sphere.

It is easily seen that this theorem is a generalization of Theorem (1.2).

Theorem (C). If a compact orientable space K_n with constant scalar curvature field R and of dimension >2 admits an infinitesiamal non–isometric conformal transformation on $v^h : \pounds_v g_{ji} = 2\rho g_{ji}$, $\rho \neq 0$. then

(1.8) $\int_{Kn} \pounds_v \pounds_v (\alpha_0 w - \alpha_1 \Delta w)\, dv \geq \frac{n\,(n+2)}{2} ,\int_{Kn} \rho^2 (\alpha_0 w - \alpha_1 \Delta w)\, dv,$

holds, where a and b are constants such that $a + b \neq 0$, α_0 and α_1, are non–negative constants not both zero, such that if $n > 6$ the first inequality in (1.4) holds, the equality in (1.8) holds if and only if K_n is isometric to a sphere.

Theorem (D). If a compact Orientable space K_n with constant scalar curvature field R and of dimension >2 admits an infinitesimal non–isometric conformal transformation $v^h : \pounds_v g_{ji} = 2\rho g_{ji}$, $\rho \neq 0$. then

(1.9) $\int_{Kn} \pounds_v \pounds_v (\alpha_0 w + \alpha_1 \Delta w)\, dv \geq 0$

holds, where a and b are constant such that $a + b \neq 0$ and α_0 and α_1 are non–negative constants, not both zero, such that the first inequality in (1.5) holds, the equality in (1.9) holds if and only if K_n is isometric to a sphere.

It can easily be seen that this theorem is a generalization of Theorem (1.3)

Remark (1.1). Making use of Lemma (2.1), which will be proved in section 2, we can easily prove that any infinitesimal homothetic transformation of a compact orientable Kaehlerian space is necessarily and infinitesimal isometric transformation.

Remark (1.2). Yano [4] proved that if a compact Orientable Riemannian space in the constant scalar curvature field R and of dimension ≥ 2 admits an infinitesimal non–isometric conformal transformation, then R is necessarily a positive constant.

2. Lemmas. In a Riemannian space M_n, for an infinitesimal conformal transformation v^h : $\pounds_v g_{ji} = 2\rho g_{ji}$, $\rho \neq 0$, we have [3]

$$\pounds_v R^h{}_{kji} = -\delta^h{}_k \nabla_j \rho_i + \delta^h{}_j \nabla_k \rho_i - (\nabla_k \rho^h) g_{ji} + (\nabla_i \rho^h) g_{ki} ,$$

$$\pounds_v R_{ji} = -(n-2) \nabla_j \rho_i - \Delta \rho\, g_{ji} ,$$

$$\pounds_v R = -2(n-1) \Delta \rho - 2\rho R .$$

Thus, for K_n with constant scalar curvature field R,

$\Delta \rho = -R\rho/(n-1)$ and

(2.1) $\nabla_j G_{ji} = 1/2\,(n-2)\, n^{-1} \nabla_i R = 0.$

We have, from (1.2),

(2.2) $\pounds_v G_{ji} = -(n-2)(\nabla_i \rho_i - \Delta \rho\, g_{ji}/n)$ and

$$\pounds_v Z_{kjih} = -g_{kh}\nabla_i \rho_i - g_{jh}\nabla_k \rho_i - (\nabla_k \rho_h) g_{ji} + (\nabla_j \rho_h) g_{ki}$$
$$+ 2\nabla\rho\,(g_{kh} g_{ji} - g_{jh} g_{ki})/n + 2\rho Z_{kjih} .$$

By strightforward calculations, we have, in view of (1.3) and (2.2),

$$(\pounds_v W_{kjih}) W^{kjih} = -4(a+b)^2 (\nabla^j \rho^i) g_{ji} + 2\rho W_{kjih} W^{kjih} .$$

On the other hand, we get

$$(\pounds_v W^{kjih}) W_{kjih} = (\pounds_v W_{kjih}) W^{kjih} - 8\rho W_{kjih} W^{kjih} .$$

Thus, we have

(2.3) $\pounds_v w = -8(a+b)^2 (\nabla^j \rho^i) G_{ji} - 4\rho w.$

Lemma (2.1). If a compact orientable Kaehlerian space K_n of dimension n admits an infinitesimal conformal transformation v^h : $\pounds_v g_{ji} = 2\rho g_{ji}$ then for any founction F on K_n,

(2.4) $\int_{Kn} \rho F dv = -1/n \int_{Kn} \pounds_v F dv.$

Proof. Since $\rho = \nabla_a v^a/n$, we have, by using Green's Theorem,

$$\int_{Kn} \nabla_a (F v^a)\, dv = \int_{Kn} \pounds_v F dv + \int_{Kn} \rho F dv = 0 ,$$

which proves (2.4).

Lemma (2.2). If a compact Orientable Kaehlerian space K_n with constant scalar curvature field R and of dimension n admits an infinitesimal conformal transoframation on v^h : $\pounds_v g_{ji} = 2\rho g_{ji}$, then $\int_{Kn} \rho\,(\nabla^j \rho^i)\, G_{ji}\, dv = -\int_{Kn} G_{ji}\,\rho^j \rho^i\, dv.$

Proof. This follows from (2.1) and

$$\int_{Kn}\nabla_j (G_{ji}\,\rho^i \rho)\, dv = \int_{Kn} (\nabla^j G_{ji})\,\rho^i \rho\, dv + \int_{Kn} G_{ji}\ (\nabla^j \rho^i)\ \rho\, dv + \int_{Kn} G_{ji}\,\rho^j \rho^i\, dv = 0.$$

Lemma (2.3).) (Hiramatu [1]): If a compact Orientable Kaehlerian space K_n with constant scalar curvature field R and of dimension n admits an infinitesimal conformal transformation v^h : $\pounds_v g_{ji} = 2\rho g_{ji}$, then

$$(2.5)\quad \int_{Kn} \pounds_v \pounds_v f\, dv = -2n\,(n-2) \int_{Kn} G_{ji}\,\rho^j \rho^i\, dv + 4n \int_{Kn} \rho^2 f\, dv,$$

$$\int_{Kn} \pounds_v \pounds_v z\, dv = -8n \int_{Kn} G_{ji}\,\rho^j \rho^i\, dv + 4n \int_{Kn} \rho^2 z\, dv.$$

Lemma (2.4). If a compact Orientable Kaehlerian space K_n with constant scalar curvature field R and of dimension n admits an infinitesimal conformal transformation v^h : $\pounds_v g_{ji} = 2\rho g_{ji}$, then

$$(2.6)\quad \int_{Kn} \pounds_v \pounds_v w\, dv = -8n\,(a+b)^2 \int_{Kn} G_{ji}\,\rho^j \rho^i\, dv + 4n \int_{Kn} \rho^2 w\, dv.$$

Proof. Making use of (2.3) and Lemmas (2.1) and (2.2), we have

$$\int_{Kn} \pounds_v \pounds_v w\, dv = -n \int_{Kn} \rho\, \pounds_v w\, dv = 8n\,(a+b)^2 \int_{Kn} \rho(\nabla^j \rho^i)\, G_{ji}\ dv + 4n \int_{Kn} \rho^2 w\, dv,$$

which shows (2.6).

Lemma (2.5). (Hiramatu [1]) : If a compact Orientable Kaehlerian space K_n with constant scalar curvature field R and of dimension $n \geq 2$ admits an infinitesimal conformal transformation v^h : $\pounds_v g_{ji} = 2\rho g_{ji}$, then for any fucntion F on K_n,

$$\int_{Kn} \pounds_v \pounds_v \nabla F\, dv = -\frac{R}{n-1} \int_{Kn} \pounds_v \pounds_v\ F\, dv + \frac{n\,(n+2)}{2} \int_{Kn} (\rho^2\, \Delta F)\, dv.$$

Lemma (2.6). (For example, Yano [5]). If a compact Orientable Kaehlerian space K_n with constant scalar curvture field R and of dimenstion >2 admits on infinitesimal non–isometric conformal transformation v^h : $\pounds_v g_{ji} = 2\rho g_{ji}$, $\rho \neq 0$, then

$$\int_{Kn} G_{ji}\,\rho^j \rho^i\, dv \leq 0,$$

the equality holds if and only if K_n is isometric to a sphere.

3. Proofs of Theorems. We shall, now, prove our theorems mentioned in section 1.

Proof of theorem(A). Making use of (2.5) in Lemmas (2.3) and (2.5), we get

$$\int_{Kn} £_v £_v (\alpha_0 f + \beta_0 z - \alpha_1 \Delta f - \beta_1 \Delta z)\, dv - \frac{n\,(n+2)}{2} \int_{Kn} \rho^2 (\alpha_0 f + \beta_0 z - \alpha_1 \Delta f - \beta_1 \Delta z)\, dv$$

$$= \int_{Kn} £_v £_v (\alpha_0 f + \beta_0 z)\, dv - \frac{R}{n-1} \int_{Kn} £_v £_v (-\alpha_1 f - \beta_1 z)\, dv + \frac{n\,(n+2)}{2} \{ \int_{Kn} \rho^2 (-\alpha_1 \Delta f - \beta_1 \Delta z)\, dv - \int_{Kn} \rho^2 (\alpha_0 f + \beta_0 z - \alpha_1 \Delta f - \beta_1 \Delta z)\, dv \}$$

$$= (\alpha_0 + \frac{R}{n-1} \alpha_1) \int_{Kn} £_v £_v f\, dv + (\beta_0 + \frac{R}{n-1} \beta_1) \int_{Kn} £_v £_v z\, dv - \frac{n\,(n+2)}{2} \{ \int_{Kn} \rho^2 (\alpha_0 f - \beta_0 z)\, dv$$

$$= -[\, 2n\,(n-2)\,(\alpha_0 + \frac{R}{n-1} \alpha_1) + 8n\,(\beta_0 + \frac{R}{n-1} \beta_1)] \int_{Kn} G_{ji} \rho^j \rho^i\, dv$$

$$+ n\,(\frac{4R}{n-1} \alpha_1 - \frac{n-6}{2} \alpha_0) \int_{Kn} \rho^2\, dv + n\,(\frac{4R}{n-1} \beta_1 - \frac{n-6}{2} \beta_0) \int_{Kn} \rho^2 z\, dv$$

From Lemma (2.6) and our assumption, we can see that the right hand side of the above relation in non–negative and consequently (1.6) holds. If the equality in (1.6) holds, then, from our assumption, we have

$$(3.1) \int_{Kn} G_{ji} \rho^j \rho^i\, dv = 0,$$

and K_n is isometric to a sphere, by virtue of Lemma (2.6).

Conversely, If K_n is isometric to a sphere, we get $G_{ji} = 0$ and $Z_{kjih} = 0$ and the equality in (1.6) holds.

Remark (3.1). If we assueme that $\alpha_0 f + \beta_0 z - \alpha_1 \Delta f - \beta_1 \Delta z = c$ (constant), from Theorem (A), we have $c \le 0$. On the other hand $c \ge 0$ holds, because

$$c \int_{Kn} dv = \int_{Kn} c\, dv = \int_{Kn} (\alpha_0 f + \beta_0 z - \alpha_1 \Delta f - \beta_1 \Delta z)\, dv = \alpha_0 \int_{Kn} f\, dv + \beta_0 \int_{Kn} z\, dv \ge 0.$$

Thus, if $\alpha_0 f + \beta_0 z - \alpha_1 \Delta f - \beta_1 \Delta z$ is a constant, then the constant is equal to zero and consequently the equality in (1.6) holds, and K_n is isometric to a sphare. This fact shows that Theorem *(A)* is genralization of Theorem (1.1).

Proof of Theorem (B). Making use of (2.5) in Lemmas (2.3), (2.5) and

(3.2) $1/2\, \nabla \rho^2 = \rho_i \rho^i - R\, \rho^2/(n-1)$,

we have

$$\int_{Kn} \pounds_v \pounds_v (\alpha_0 f + \beta_0 z + \alpha_1 \Delta f + \beta_1 \Delta z)\, dv = (\alpha_0 - \frac{R}{n-1}\alpha_1) \int_{Kn} \pounds_v \pounds_v f\, dv + (\beta_0 - \frac{R}{n-1}\beta_1) \int_{Kn} \pounds_v \pounds_v z\, dv + \frac{n(n+2)}{2}\, \alpha_1 \int_{Kn} (\Delta\rho^2) f\, dv + \frac{n(n+2)}{2}\, \beta_1 \int_{Kn} (\Delta\rho^2) z\, dv$$

$$= (\alpha_0 - \frac{R}{n-1}\alpha_1) \int_{Kn} \pounds_v \pounds_v f\, dv + (\beta_0 - \frac{R}{n-1}\beta_1) \int_{Kn} \pounds_v \pounds_v z\, dv + n(n+2) \int_{Kn} \rho_i \rho^i (\alpha_1 f + \beta_1 z)\, dv + n(n+2) \frac{R}{n-1} \int_{Kn} \rho^2 (\alpha_1 f + \beta_1 z)\, dv$$

$$= -[\, 2n(n-2)\, (\alpha_0 - \frac{R}{n-1}\alpha_1) + 8n\, (\beta_0 - \frac{R}{n-1}\beta_1) \int_{Kn} G_{ji} \rho^j \rho^i\, dv + n(n-2) \int_{Kn} \rho_i \rho^i (\alpha_1 f + \beta_1 z)\, dv + n\, [\, 4\alpha_0 - \frac{(n+6)R}{n-1}\alpha_1] \int_{Kn} \rho^2 f\, dv + n\, [\, 4\beta_0 - \frac{(n+6)R}{n-1}\beta_1] \int_{Kn} \rho^2 z\, dv .$$

From Lemma (2.6) and our assumption, we can see that the right hand side of the above equation is non–negative and consequently (1.7) holds, because it follows from our assumption that

$$\alpha_0 - R\,(n-1)^{-1} \alpha_1 \text{ and } \beta_0 - R\,(n-1)^{-1} \beta_1$$

are non–negative and not both zero. If the equality in (1.7) holds, then we have (3.1) and K_n is isometric to a sphere by virtue of Lemma (2.6)

Conversely, If Kn is isometric to a shpere, we have $G_{ji} = 0$ and $Z_{Kjih} = 0$ and the equality in (1.7) holds.

Proof of Theorem (C). Similarly, as in th proof of the Theorem (A), by using (2.6) in Lemma (2.4) and Lemmas (2.5) and (2.6), we get

$$\int_{Kn} \pounds_v \pounds_v (\alpha_0 w - \alpha_1 \Delta w)\, dv - \frac{n(n+2)}{2}\, , \int_{Kn} \rho^2 (\alpha_0 w - \alpha_1 \Delta w)\, dv,$$

$$= (\alpha_0 + \frac{R}{n-1}\alpha_1) \int_{Kn} £_v £_v w dv - \frac{n(n+2)}{2}\alpha_0) \int_{Kn} \rho^2 w\, dv$$

$$= -8n\,(a+b)^2\ (\alpha_0 + \frac{R}{n-1}\alpha_1) \int_{Kn} G_{ji}\rho^j \rho^i\, dv + n\,(\frac{4R}{n-1}\alpha_1 - \frac{n-6}{2}\alpha_0)$$

$$\int_{Kn} \rho^2 w\, dv \geq 0,$$

which proves (1.8). It is easily proved from Lemma (2.6) and our assumption that the equality in (1.8) holds if and only if K_n is isometric to a spere.

Proof of Theorem (D). Similarly, as in the proof of Theorem (B), by using (2.6) in Lemmas (2.4), (2.5) and (2.6) and (3.2), we have

$$\int_{Kn} £_v £_v (\alpha_0 w + \alpha_1 \Delta w)\, dv$$

$$= -8n\,(a+b)^2\ (\alpha_0 - \frac{R}{n-1}\alpha_1) \int_{Kn} g_{ji}\rho^j \rho^i\, dv + n\,(n+2)\,\alpha_1 \int_{Kn} \rho_i \rho^i w\, dv$$

$$+ n\ [4\alpha_0 - \frac{(n+6)R}{n-1}\alpha_1] \int_{Kn} \rho^2 w\, dv \geq 0,$$

which proves (1.9). It is easily proved from Lemma (2.6) and our assumption that the equality in (1.9) holds if and only if K_n is isometric to a sphere.

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