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SOME NEW LINEAR AND BILINEAR GENERATING RELATIONS INVOLVING HYPERGEOMETRIC FUNCTIONS OF TWO AND THREE VARIABLES

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ABSTRACT

The aim of this paper is to establish some linear and bilinear generating relations involving hypergeometric functions of two and three variables. Some specializations, relevant to the present discussion, are also discussed.

1. Introduction. The hypergeometric functions of three variables G_A, G_B defined by Pandey [2] and Horn's functions of two variables H_2, H_3 as given in [1, p. 224-226], are as follows :

$$G_A(\alpha,\beta,\beta';\gamma\,;\,x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(\alpha,n+p-m)\,(\beta,m+p)\,(\beta',n)}{(1,m)\,(1,n)\,(1,p)\,(\gamma,n+p-m)}\, x^m y^n z^p\,, \qquad \ldots(1.1)$$

$$G_B(\alpha,\beta_1,\beta_2,\beta_3\,;\,\gamma\,;\,x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(\alpha,n+p-m)\,(\beta_1,m)\,(\beta_2,n)\,(\beta_3,p)}{(1,m)\,(1,n)\,(1,p)\,(\gamma,n+p-m)}\, x^m y^n z^p, \qquad \ldots(1.2)$$

$$H_2(\alpha,\beta,\gamma,\delta;\epsilon\,;\,x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha,m-n)\,(\beta,m)\,(\gamma,n)\,(\delta,n)}{(1,m)\,(1,n)\,(\epsilon,m)}\, x^m y^n\,, \qquad \ldots(1.3)$$

$$H_3(\alpha,\beta\,;\,\gamma\,;\,x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha,m-n)\,(\beta,m)}{(1,m)\,(1,n)\,(\gamma,m)}\, x^m y^n\,, \qquad \ldots(1.4)$$

In our investigations, we also require the following relations [3, p. 153, (2.1), (2.2) and p. 154, (3.1) , (3.2)];

$$(1-x)^{1-\alpha-\delta} H_2 \left(1-\alpha-\delta,\upsilon,\gamma,\beta;\upsilon; x, \frac{yt}{(1-x)(1-y+yt)}\right)$$

$$= \frac{\Gamma(\delta+\alpha)\,\Gamma(\delta-\gamma)}{\Gamma(\delta)\,\Gamma(\delta+\alpha-\gamma)} t^{-\alpha} (1-y+yt)^{\beta} \sum_{n=0}^{\infty} \frac{(1,n)}{(\delta,n)} (y-1)^n$$

$$P_n^{(\alpha-n,\beta+n)} \left(\frac{1+y}{1-y}\right) \; P_n^{(\delta-1,-\gamma-n)} (1-2t), \qquad \ldots (1.5)$$

$$(1-t)^{-1-\alpha} \exp\left(\frac{-t\,(x+y)}{1-t}\right) (1+x)^{-\alpha} H_3 \left(-\alpha,\beta;\beta; x, \frac{-xyt}{(1-x)(1-t)^2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(1,\, n)}{(1+\alpha,n)} t^n L_n^{(\alpha)}(x)\, L_n^{(\alpha)}(y), \qquad \ldots (1.6)$$

$$G_A \left(\alpha,\beta_1,\beta_2\,;\, \alpha\,;\, vt, v(1-2t),\, v\,(1-2t)\right)$$

$$= \sum_{n=0}^{\infty} \frac{(1,\, n)}{(\beta_1+\beta_2,n)} (v-1)^n P_n^{(-n,\beta_1+\beta_2+n)} \left(\frac{1+v}{1-v}\right) P_n^{(\beta_1+\beta_2-1,-\beta_2-n)} (1-2t) \qquad \ldots (1.7)$$

$$G_B \left(\alpha,\beta_1,\beta_2\,,\beta_3;\, \alpha\,;\, vt, v(2t-1),\, v\,(2t-1)\right)$$

$$= \sum_{n=0}^{\infty} \frac{(1,\, n)}{(-\beta_2-\beta_3,n)} (v-1)^n P_n^{(-n,1-\beta_2-\beta_3+n)} \left(\frac{1+v}{1-v}\right)$$

$$P_n^{(-\beta_2-\beta_3\,,\, \beta_1+\beta_2+\beta_3-1-n)}(1-2t)\,. \qquad \ldots (1.8)$$

In the present paper, we shall establish linear and bilinear genrating relations involving above hypergeometric functions of two and three variables, Some special cases will also be discussed.

2. Bilinear Generating Relations. We establish here the following bilinear generating relations :

$$\sum_{r=0}^{\infty} \frac{(\upsilon,r)}{(1,r)} (1-x)^{1-\alpha-\delta} H_2\left(1-\alpha-\delta,\upsilon+r,\gamma,\beta;\upsilon; x\,(1+z), \frac{yt}{(1-x)(1-y+yt)}\right) (-z)^r$$

$$= (1+z)^{-\upsilon} \frac{\Gamma(\delta+\alpha)\,\Gamma(\delta-\gamma)}{\Gamma(\delta)\,\Gamma(\delta+\alpha-\gamma)} t^{-\alpha} (1-y+yt)^{\beta} \sum_{n=0}^{\infty} \frac{(1,n)}{(\delta,n)} (y-1)^n$$

$$P_n^{(\alpha-n,\beta+n)} \left(\frac{1+y}{1-y}\right) \; P_n^{(\delta-1,-\gamma-n)} (1-2t) \quad , \qquad \ldots (2.1)$$

$$\sum_{r=0}^{\infty} \frac{(\beta,r)}{(1,r)} (1-x)^{-1-\alpha} \exp\left(\frac{-t\,(x+y)}{1-t}\right) (1+x)^{-\alpha}$$

$$H_3 \left(-\alpha,\beta+r;\, \beta;\, x\,(1+z), \frac{-xyt}{(1-x)(1-t)^2}\right) (-z)^r$$

$$= (1+z)^{-\beta} \sum_{n=0}^{\infty} \frac{(1,\, n)}{(1+\alpha,n)} t^n L_n^{(\alpha)}(x)\, L_n^{(\alpha)}(y) \quad , \qquad \ldots (2.2)$$

$$\sum_{r=0}^{\infty} \frac{(\beta_2,r)}{(1,r)} G_A\,(\alpha,\beta_1,\beta_2+r,\alpha\,;\,vt,v(1-2t)\,(1+z)\,,\,v(1-2t))\,(-z)^r$$
$$= (1+z)^{-\beta_2} \sum_{n=0}^{\infty} \frac{(1,\,n)}{(\beta_1+\beta_2\,,n)}\,(v-1)^n\, P_n^{(-n,\beta_1+\beta_2+n)}\,(\frac{1+v}{1-v})$$
$$P_n^{(\beta_1+\beta_2-1,-\beta_2-n)}\,(1-2t)\,, \qquad \ldots(2.3)$$

$$\sum_{r=0}^{\infty} \frac{(\beta_2,r)}{(1,r)}\, G_B\,(\alpha,\beta_1,\beta_2+r,\beta_3\,;\,\alpha\,;\,vt,v(2t-1)\,(1+z),\,v\,(2t-1))\,(-z)^r$$
$$= (1+z)^{-\beta_2} \sum_{n=0}^{\infty} \frac{(1,\,n)}{(1-\beta_2-\beta_3\,,n)}\,(v-1)^n\, P_n^{(-n,1-\beta_2-\beta_3+n)}\,(\frac{1+v}{1-v})$$
$$P_n^{(-\beta_2-\beta_3\,,\,\beta_1+\beta_2+\beta_3-1-n)}(1-2t)\,. \qquad \ldots(2.4)$$

Derivation of (2.1) . Consider

$$T = \sum_{r=0}^{\infty} \frac{(\upsilon,r)}{(1,r)}(1-x)^{1-\alpha-\delta}\, H_2(1-\alpha-\delta,\upsilon+r,\gamma,\beta;\,\upsilon;\,x\,(1+z),\, \frac{yt}{(1-x)\,(1-y+yt)})\,(-z)^r.$$

On expressing H_2 in series form, we have

$$T = \sum_{r=0}^{\infty} \frac{(\upsilon,r)}{(1,r)}\,(1-x)^{1-\alpha-\delta} \sum_{m,n=0}^{\infty} \frac{(1-\alpha-\delta,m-n)\,(\upsilon+r,\,m)\,(\gamma,n)\,(\beta\,,n)}{(1,m)\,(1,n)\,(\upsilon,m)}$$
$$\{x(1+z)\}^m\{\frac{yt}{(1-x)\,(1-y+yt)}\}^n\,(-z)^r.$$

Again using the results

$$(a,n+k) = (a,n)\,(a+n,k) = (a,k)\,(a+k,n)\,, \qquad \ldots\ (2.5)$$

and

$$\sum_{n=0}^{\infty} \frac{(\alpha,r)}{(1,r)}\,(-t)^n = (1+t)^{-\alpha}, \qquad \ldots\ (2.6)$$

we get

$$T = (1-x)^{-\upsilon}\,(1-x)^{1-\alpha-\delta} \sum_{m,n=0}^{\infty} \frac{(1-\alpha-\delta,m-n)\,(\upsilon,m)\,(\gamma,n)\,(\beta\,,n)}{(1,m)\,(1,n)\,(\upsilon,m)}\,.$$
$$x^m\{\frac{yt}{(1-x)\,(1-y+yt)}\}^n.$$

Now applying the result (1.3), we obtain

$$T = (1+z)^{-\upsilon}\,(1-x)^{1-\alpha-\delta}\, H_2(1-\alpha-\delta,\upsilon,\gamma,\beta;\,\upsilon;\,x,\frac{yt}{(1-x)\,(1-y+yt)})\,.$$

which, in light of (1.5), provides (2.1).

The proof of the formula (2.2) to (2.4) would run parallel to what we have obtained above in view of (1.6), (1.7) and (1.8) respectively.

3. Linear Generating Relations. In this section we establish the following linear generating relations:

$$\sum_{n=0}^{\infty} \frac{(\gamma,n)}{(1,n)} H_2(\alpha,\beta,\gamma+n,\delta;\in;x,y)\,t^n$$
$$= (1-t)^{-\gamma} H_2(\alpha,\beta,\gamma,\delta;\in;x,\frac{y}{1-t}), \qquad ...(3.1)$$
$$\sum_{n=0}^{\infty} \frac{(\beta,n)}{(1,n)} H_3(\alpha,\beta+n,\gamma;\ x,y)\,t^n$$
$$= (1-t)^{-\beta} H_3(\alpha,\beta,\gamma;\ \frac{x}{1-t},y), \qquad ...(3.2)$$
$$\sum_{n=0}^{\infty} \frac{(\beta',n)}{(1,n)} G_A(\alpha,\beta,\beta'+n,\gamma\ ;x,y,z)\,t^n$$
$$= (1-t)^{-\beta'} G_A(\alpha,\beta,\beta';\gamma\ ;x,\frac{y}{1-t},z), \qquad ...(3.3)$$
$$\sum_{n=0}^{\infty} \frac{(\beta_1,n)}{(1,n)} G_B(\alpha,\beta_1+n,\beta_2,\beta_3\ ;\gamma\ ;x,y,z)\,t^n$$
$$= (1-t)^{-\beta_1} G_B(\alpha,\beta_1,\beta_2,\beta_3\ ;\gamma\ ;\frac{x}{1-t},y,z), \qquad ...(3.4)$$

Derivation of (3.1) . Consider

$$\Delta = \sum_{n=0}^{\infty} \frac{(\gamma,n)}{(1,n)} H_2(\alpha,\beta,\gamma+n,\delta;\in;x\,,y\,)\,t^n$$

On expressing H_2 in series form, we get

$$\Delta = \sum_{n=0}^{\infty} \frac{(\gamma,n)}{(1,n)} \sum_{m,n=0}^{\infty} \frac{(\alpha,r-s)\,(\beta\,,r)\,(\gamma+n,\,s)\,(\delta,s)}{(1,r)\,(1,s)\,(\in,r)} x^r y^s\,t^n$$

Again using the results (2.5) and (2.6), we have

$$\Delta = (1-t)^{-\gamma} \sum_{r,s=0}^{\infty} \frac{(\alpha,r-s)\,(\beta\,,r)\,(\gamma,\,s)\,(\delta,s)}{(1,r)\,(1,s)\,(\in,r)} x^r \left(\frac{y}{1-t}\right)^s,$$

which, in light of (1.3), provides (3.1).

Proceeding on similar lines as above results (3.2) to (3.4) can be derived.

3. Special Case.

(i) On taking $y = 0$ in (2.1) and simplifying, we obtain

$$(1-x)^{1-\alpha-\delta}\ {}_2F_1\left[\begin{matrix}1-\alpha-\delta,\upsilon: \\ \upsilon:\end{matrix}\ x\right]$$
$$= \frac{\Gamma(\delta+\alpha)\,\Gamma(\delta-\gamma)}{\Gamma(\delta)\,\Gamma(\delta+\alpha-\gamma)} t^{-\alpha} \sum_{n=0}^{\infty} (-1)^n \frac{(1+\alpha-n,n)}{(\delta,n)} P_n^{(\delta-1,-\gamma-n)}(1-2t), \qquad (4.1)$$

(ii) In (2.3) putting $t=0$, $\beta_1= 0$ and solving, we find that

$$ {}_2F_1\left[\begin{matrix}\alpha,\beta_2\,; \\ \alpha;\end{matrix}\ \upsilon\right] = \sum_{n=0}^{\infty} (\upsilon-1)^n P_n^{(-n,\,\beta_2+n)}\left(\frac{1+\upsilon}{1-\upsilon}\right). \qquad ...(4.2)$$

which is a result due to Singh [4, p. 78. (4.2)].

(iii) In (2.4) setting $t=0$, $\beta_1 = 0$, replacing v by $-v$ and simplifying, we get

$$_2F_1\ [\ \alpha, \beta_2 + \beta_3 ; \alpha\ ; v] = \sum_{n=0}^{\infty} \{-(v+1)\}^n\ P_n^{(-n,\ 1-\beta_2-\beta_3+n)}\ (\frac{1+v}{1-v}). \quad ...(4.3)$$

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REFERENCES

[1] A. Erdélyi, *Higher Transcendental Functions,* Vol-I McGraw-Hill, New York, 1953.

[2] R.C. Pandey, On certain hypergeometric transformation, *J. Math. and Mech.* **12** (1963), 113-118.

[3] B.M. Srivastava, Bilinear generating relations for Jacobi and Laguerre polynominals, *Vidya B* **20** (1977), 150-156.

[4] M.B. Singh, and B.M.L. Srivastava, Some linear and bilinear generating relations involving hypergeometric functions of three variables, *Jñānābha* , **16** (1986), 67-80.