Jñānābha, Vol. 28, 1998

GENERAL FRACTIONAL INTEGRAL FORMULAS INVOLVING THE GENERALIZED POLYNOMIALS SETS AND FOX'S *H*-FUNCTION

By

R. K. Laddha

Department of Mathematics

M.L.V. Government College, Bhilwara – 311001, Rajasthan, India

(Received : September 24, 1997)

ABSTRACT

In this paper we obtain generalized fractional integral formulas involving two classes of generalized polynomials sets and Fox's *H*-function. On account of the general nature of fractional ingegral operatiors (*FIO*) and Fox's *H*-function, our findings provide integresting unifications nad extension of number of (known and new) results. Some interesting special cases of our main results have been mentioned briefly.

1. Introduction The aim of this paper is to study the following pair of fractional integral operatiors (*FIO*) :

$$R\,[f(x)] = r\,x^{-\eta-r\alpha-1}\int_0^x t^{\eta}\,(x^r-t^r)^{\alpha}\,S_n^{\alpha,\beta,\tau}\,[e\,(t^r/x^r)^{\rho}\,(1-t^r/x^r)^{\sigma}]\,f(t)\,dt \quad ...(1.1)$$

and

$$K\,[f(x)] = r\,x^{-\delta}\int_x^{\infty} t^{\delta-r\alpha-1}(t^r-x^r)^{\alpha}\,S_n^{\alpha,\beta,\tau}\,[e(t^r/x^r)^{\rho'}(1-t^r/x^r)^{\sigma'}]\,f(t)\,dt \quad ...(1.2)$$

where $S_n^{\alpha,\beta,\tau}\,[x]$ stands for generalized polynomials set. The explicit form of this generalized polynomial set (Raizada [3, p. 71, (2.3.4)] see also, Saigo et al [4]) is :

$$S_n^{\alpha,\beta,\tau}\,[x\,;p,\,s,\,q,\,A,\,B,\,m,\,k,\,l]$$

$$= B^{qn}\,x^{l(m+n)}\,(1-\tau x^p)^{sn}\,l^{m+n}\sum_{v=0}^{m+n}\sum_{u=0}^{v}\sum_{j=0}^{m+n}\sum_{w=0}^{j}$$

$$\frac{(-1)^j\,(-j)_w\,(\alpha)_j\,(-v)_u\,(-\alpha-qn)_w}{w\,!\,j!\,u!\,v!\,(1-\alpha-j)_w}\,(-\beta/\tau-sn)_v\,\left(\frac{w+k+pu}{l}\right)_{m+n}$$

$$\left(\frac{-\tau x^p}{1-\tau x^p}\right)^v\left(\frac{Ax}{B}\right)^j \quad ...(1.3)$$

It may be pointed out here that polynomial set defined by (1.3) is very general in nature and it unifies and extends a number of classical polynomials introduced and studied by various research workers from time to time. Some special cases of (1.3) are given by Raizada [(3,p. 65] in tabular form, see also Saigo et al [4]).

Throughout this paper we assume that $f(t) \in A$, where A denotes the class of functions $f(t)$ for which $\int f(t)\, dt < \infty$ for every closed interval $[a, b]$ excluding the origin and

$$f(t) = \begin{cases} 0(|t|^{M}) & \text{for small values of } t \\ 0(|t|^{-N} e^{-Q|t|}) & \text{for large values of } t \end{cases}$$

If $f(t) \in A$, then the operators defined by (1.1) exists if

(i) $|\tau x^r| < 1,\ r \in N,$

ρ, σ are non-neagative real numbers (not both zero simultaneously)

(ii) $\text{Re}\,[\eta + M + r\,\rho\,(pv + j + l\,(m+n))] > -1,$

$\text{Re}\,[\,r\,\alpha + M + r\,\sigma(pv+j + l\,(m+n))] > -1$

and the operator defined by (1.2) exists if

(i) $|\tau x^r| < 1,\ r \in N,$

(ii) $\text{Re}\,[\alpha + \sigma' r\ (pv + j + l\,(m+n))] > -1,$

ρ', σ' are non-neagative real numbers(not both zero simultaneously)

(iii) $\text{Re}\,(Q) > 0$ or $\text{Re}\,(Q) = 0$ with

$\text{Re}\,[\rho' r\ (pv + j + l\,(m+n)) + N - \delta] > -2,$

$v, j = 0, 1, \ldots, m + n$

the *FIO* defined by (1.1) and (1.2) provide us generalization of the well-known *FIO* introduced by Kober, Riemann–Lioubille, Weyl and many other research workers from time to time.

2. Main Results

I.

$$R\left[t^{\lambda} (ct^{\mu}+d)^{\upsilon}\, H^{m_1,n_1}_{p_1,q_1}\left[zt^{\lambda_1} (ct^{\mu}+d)^{\upsilon_1} \middle| \begin{matrix} (a_j, \alpha_j)_{1,p_1} \\ (b_j, \beta_j)_{1,q_1} \end{matrix} \right] S^{m_2}_{n_2}\left[yt^{\lambda_2} (ct^{\mu}+d)^{\upsilon_2} \right] \right].$$

$$= \sum_{v=0}^{m+n} \sum_{u=0}^{v} \sum_{j=0}^{m+n} \sum_{w=0}^{j} \theta(w, j, u, v) \sum_{g=0}^{\infty} \sum_{j=0}^{[n_2/m_2]} \sum_{i=0}^{\infty}$$

$$\frac{(v-sn)_g}{g!}\tau^g e^{E+pg}\frac{(-n_2)_{m_2h}}{h!}A_{n_2,h}\,y^h\frac{c^i d^{-i+\upsilon+\upsilon_2 h}}{i!}x^{\lambda_2 h+\lambda+\mu i}\,\Gamma(\alpha+\sigma(E+pg)+1)$$

$$H^{m_1,n_1+2}_{p_1+2,\,q_1+2}\left[z\,x^{\lambda_1}d^{\upsilon_1}\left|\begin{matrix}(1-\eta+\lambda+\rho r(E+pg)+\lambda_2 h+\mu i+1)/r),(\lambda_1/r)\\(b_j,\beta_j)_{1,q_1}(i-\upsilon-\upsilon_2 h,\upsilon_1),\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(a_j,\alpha_j)_{1,p_1},\\(-[\eta+\lambda+\lambda_2 h+\mu i+1)/r]-\alpha-(\rho+\sigma)(E+pg),\lambda_1/r)\end{matrix}\right] \quad \text{... (2.1)}$$

where $E = l\,(m+n) + pv + j,\ v\,,\ j = 0,\ 1,\ ...,\ m+n$ ... (2.2)

$$\theta(w,j,u,v) = B^{qn}\, l^{m+n}\,\frac{(-1)^j\,(-v)_w\,(\alpha)_j\,(-v)_u\,(-\alpha-qn)_w}{w\,!\,j!\,u!\,v!\,(1-\alpha-j)_w}\,(-\beta/\tau-sn)_v$$

$$((k+pu+w)/\rho)_{m+n}\quad(A/B)^j\,(-\tau)^v \quad \text{...(2.3)}$$

and

$$S_n^{m+n}[x] = \sum_{n=0}^{[n/m]}\frac{(-n)_{mh}}{h!}A_{n,h}x^h, \quad \text{...(2.4)}$$

stands for general class of polynomials defined by Srivastava ([5,p.1], see also Srivastava and Singh [7]). Here coefficients $A_{n,h}$ $(n,\ h \geq 0)$ are arbitrary constants, real or complex, and m is a positive integer.

Also $H^{m,n}_{p,q}[x]$ stands for well-known Fox's H-function.

For details for this function see e.g. [6].

Formula (2.1) is valid under the following conditions :

(i) $|\tau x^r| < 1,\ r \in N,\ min\ \{\mu,\rho,\sigma,\lambda_1,\upsilon_1,\lambda_2,\upsilon_2\} \geq 0$

(ii) $\mathrm{Re}\,(\eta+\lambda) + r\,\rho\,E + \lambda_1 \min_{1\leq j\leq m} \mathrm{Re}\,(b_j/\beta_j) + \lambda_2 h > 0$

$\mathrm{Re}\,(\alpha) + r\,\sigma E + 1 > 0,\ h = 0,1,...,[n_2/m_2]$

(iii) $A > 0\,|\arg z\,| < 1/2A\pi$ where

$$A = \sum_{j=1}^{n_1}\beta_j - \sum_{j=n_1+1}^{p_1}\beta_j + \sum_{j=1}^{m_1}\alpha_j - \sum_{j=m_1+1}^{q_1}\alpha_j$$

(iv) m_2 is a positive integer and coefficients $A_{n_2,h}$ $(n_2,\ h \geq 0)$ are arbitrary constants, real or complex.

(v) Multiple series on right-hand side of (2.1) is absolutely convergent.

Proof: In view of definition (1.1), we have

$$R\,[f(x)] = r\,x^{-\eta-r\alpha-1}\int_0^x t^\eta\,(x^r-t^r)^\alpha\, S_n^{\alpha,\beta,\tau}\,[e\,(t^r/x^r)^\rho\,(1-t^r/x^r)^\sigma]$$

$$t^{\lambda}(ct^{\mu}+d)^{\upsilon} H^{m_1,n_1}_{p_1,q_1}\left[zt^{\lambda_1}(ct^{\mu}+d)^{\upsilon_1} \middle| \begin{matrix}(a_j,\alpha_j)_{1,p_1}\\ (b_j,\beta_j)_{1,q_1}\end{matrix}\right]$$
$$S^{m_2}_{n_2}\left[y\ t^{\lambda_2}(ct^{\mu}+d)^{\upsilon_2}\right] dt. \qquad ...(2.5)$$

Using series expasions for $S^{\alpha,\beta,\tau}_{n}$ [.] and $S^{m_2}_{n_2}$ [.] polynomials given by (1.3) and (2.4) respectively, and expressing H-function in terms of Mellin-Barnes integral [6,p. 10, (2.1.1)] and changing the order of integration and summation (which is permissible under the cnditions stated with (2.1)), we find that

$$R\,[f(x)] = r\,x^{-\eta-r\alpha-1} \sum_{v=0}^{m+n}\ \sum_{u=0}^{v}\ \sum_{j=0}^{m+n}\ \sum_{w=0}^{j}\ \theta(w,j,u,v)\,e^{E}$$
$$x^{-r(\rho+\sigma)E} \sum_{n=0}^{[n_2/m_2]} \frac{(-n_2)_{m_2 h}}{h!} A_{n_2,h}\,y^{h},$$
$$\frac{1}{2\pi w}\int_L \phi(\xi)\,w\,z^{\xi}\left[\int_0^x t^{\eta+\lambda+r\rho E+\lambda_2 h+\lambda_1\xi}\right.$$
$$\left.(x^r-t^r)^{\alpha+\sigma E}\,[1-\tau e^{\rho}(t^r/x^r)^{\rho p}\,(1-t^r/x^r)^{\sigma p}]^{sn-v}\,(ct^{\mu}+d)^{\upsilon+\upsilon_1\xi+\upsilon_2 h}\,dt\right]d\xi \qquad ...(2.6)$$

where $\phi(\xi)$ is defined in [6,p.11, (2.1.3)].

Now using following well-known binomial expansion formulas :

$$(1-z)^{-a} = \sum_{g=0}^{\infty} \frac{(a)_g}{g!} z^g,\ |z|<1, \qquad ...\ (2.7)$$

and

$$(ct^{\mu}+d)^{\upsilon} = d^{\upsilon}\sum_{i=0}^{\infty}\binom{\upsilon}{i}(ct^{\mu}/d)^{i},\ \ ct^{\mu}/d<1 \qquad ...(2.8)$$

In (2.6) changing the order of integration and summation there in, we find that

$$R\,[f(x)] = r\,x^{-\eta-r\alpha-1} \sum_{v=0}^{m+n}\ \sum_{u=0}^{v}\ \sum_{j=0}^{m+n}\ \sum_{w=0}^{j}\ \theta(w,j,u,v)$$
$$\sum_{g=0}^{\infty}\ \sum_{h=0}^{[n_2/m_2]}\ \sum_{i=0}^{\infty} \frac{(v-sn)_g}{g!}\,\tau^{g}\,e^{E+pg}\,\frac{(-n_2)_{m_2 h}}{h!}\,A_{n_2,h}\,y^{h}\,\frac{c^{i}d^{-i+\upsilon+\upsilon_2 h}}{i\,!}$$
$$x^{-r(\rho+\sigma)(E+pg)}\ 1/(2\pi w)\int_L \phi(\xi)\,z^{\xi}\,d^{\upsilon_1\xi}$$
$$\left[\int_0^x t^{\eta+\lambda+r\rho(E+pg)+\lambda_1\xi+\lambda_2 h+\mu i}\,(x^r-t^r)^{\alpha+\sigma(E+pg)}dt\right]\frac{\Gamma(\upsilon+\upsilon_1\xi+\upsilon_2 h+1)}{\Gamma(\upsilon+\upsilon_1\xi+\upsilon_2 h-i+1)}\,d\xi. \quad (2.9)$$

Evaluating the inner Beta integral occurring in (2.9) and then interpreting the result, so obtained in terms of the H-function, we get the desired result (2.1).

II

$$K\left[t^{\lambda} (ct^{\mu}+d)^{\upsilon} H^{m_1,n_1}_{p_1,q_1}\left[zt^{\lambda_1} (ct^{\mu}+d)^{\upsilon_1} \middle| \begin{matrix} (a_j,\alpha_j)_{1,p_1} \\ (b_j,\beta_j)_{1,q_1} \end{matrix} \right] S^{m_2}_{n_2}\left[y\ t^{\lambda_2} (ct^{\mu}+d)^{\upsilon_2} \right] \right].$$

$$= \sum_{v=0}^{m+n} \sum_{u=0}^{v} \sum_{j=0}^{m+n} \sum_{w=0}^{j} \theta(w,j,u,v) \sum_{g=0}^{\infty} \sum_{j=0}^{[n_2/m_2]} \sum_{i=0}^{\infty}$$

$$\frac{(\upsilon-sn)_g}{g!} \tau^g e^{E+pg} \frac{(-n_2)_{m_2 h}}{h!} A_{n_2,h}\, y^h \frac{c^i d^{-i+\upsilon+\upsilon_2 h}}{i!} x^{\lambda_2 h+\lambda+\mu i}\, \Gamma(\alpha+\sigma'(E+pg)+1)$$

$$H^{m_1+1,n_1+1}_{p_1+2,q_1+2}\left[z\, x^{\lambda_1} d^{\upsilon_1} \middle| \begin{matrix} (-\upsilon-\upsilon_2 h, \upsilon_1), (a_j,\alpha_j)_{1,p_1}, \left(\frac{-\delta+\lambda+\lambda_2 h+\mu i-r\rho'(E+pg)}{r}, \lambda_1/r\right), \\ (1-[\delta+\lambda+\lambda_2 h+ui-r\{\alpha+(\rho'+\sigma')(E+pg)\}]/r, \lambda_1/r) \\ (b_j,\beta_j)_{1,q_1} (i-\upsilon-\upsilon_2 h, \upsilon_1) \end{matrix} \right], \quad \ldots (2.10)$$

where E, $\theta(w, j, u, v)$ $S^{m_2}_{n_2}$ [.] and $H^{m_1,n_1}_{p_1,q_1}$ [.] have same meaning as mentioned with (2.1) and provided that

(i) $|\tau e^r| < 1$, $r \in N$, $\min\{\rho', \sigma', \lambda_1, \upsilon_1, \lambda_2, \upsilon_2\} \geq 0$, $\mu \leq 0$.

(ii) $Re(\square\delta - \lambda + r\rho' E - \lambda_1 h + \lambda_1 \min_{1\leq j\leq m} Re[(1-a'_j)/\alpha'_j]) > 2$

$Re(\alpha + \sigma' E) > -1$, $h = 0,1,\ldots,[n_2/m_2]$.

(iii) $A > 0\,|\arg z\,| < 1/2A\pi$

where

$$A = \sum_{j=1}^{n_1} \beta_j - \sum_{j=n_1+1}^{p_1} \beta_j + \sum_{j=1}^{m_1} \alpha_j - \sum_{j=n_1+1}^{q_1} \alpha_j$$

(iv) m_2 is a positive integer and coefficients $A_{n_2,h}$ are arbitrary constants, real or complex.

(v) The multiple series on right-hand side of (2.10) is absolutely convergent.

Proof. The result (2.10) can be proved in a manner similar to result (2.1).

3. Special Cases. The importance of our generalized fractional formulas (2.1) and (2.10) lie in their manifold generality. Firstly these results involve generalized *FIO* defined by (1.1) and (1.2). Secondly the general class of polynomials and Fox's *H*-function occurring in (2.1) and (2.10) can be suitably specialized to a remarkably wide variety of polynomials and useful special functions. Our main results thus provide unifications and extensions of various (known or new) results on *FIO*. For the sake of illustration, we mention the following interesting special cases of (2.1) only :

(i) If we take $n_2 = 0$, $A_{0,0} = 1$ in (2.1), then $S_{n_2}^{m_2}[.] = 1$ and we arrive at the following result :

(iii) Again taking $\upsilon = \upsilon_1 = \upsilon_2 = 0$ in (2.1), we get

$$R\left[t^{\lambda} H_{p_1, q_1}^{m_1, n_1}\left[zt^{\lambda_1}\left|\begin{matrix}(a_j, \alpha_j)\\(b_j, \beta_j)\end{matrix}\right.\right] S_{n_2}^{m_2}\left[yt^{\lambda_2}\right]\right]$$

$$= \sum_{v=0}^{m+n} \sum_{u=0}^{v} \sum_{j=0}^{m+n} \sum_{w=0}^{j} \theta(w, j, u, v) \sum_{g=0}^{\infty} \sum_{h=0}^{[n_2/m_2]}$$

$$\frac{(v-sn)_g}{g!} \tau^g e^{E+pg} \frac{(-n_2)_{m_2 h}}{h!} A_{n_2, h}\, y^h x^{\lambda+\lambda_2 h} \Gamma(\alpha+\sigma(E+pg)+1)$$

$$H_{p_1+1, q_1+1}^{m_1, n_1+1}\left[z x^{\lambda_1}\left|\begin{matrix}(1-\{\eta+\lambda+\rho r(E+pg)+\lambda_2 h+1\}/r), \lambda_1/r]\ (a_j, \alpha_j)_{1,p_1}\\(b_j, \beta_j)_{1,q_1}(-(\eta+\lambda+\lambda_2 h+1)/r-\alpha-(\rho+\sigma)(E+pg), \lambda_1/r]\end{matrix}\right.\right] \quad \ldots (3.3)$$

Further taking $r = 1$, $\eta, \beta \to 0$ in the above result, we obtain a recent result due to Agrawal et al [1,p. 58, Eqn. (3.1)].

(iv) Lastly taking $A = 1$, $B = s = q = m = k = 0$, $l = -1$ in (1.3), letting $\tau \to 0$ therin and using the following result [4, Eqn. (1.9)]:

$$S_n^{\alpha, \beta, 0}[x; p, 0,0,1, 0,0,0, -1]$$

$$= x^{-n}(-1)^n \sum_{v=0}^{n} \sum_{u=0}^{v} \frac{(-v)_u}{v!\,u!}(-\alpha - pu)_n (\beta x^p)^v \quad \ldots (3.4)$$

therein, we arrive at the following interesting result

$$R^*\left[t^{\lambda}(ct^{\mu}+d)^{\upsilon} H_{p_1, q_1}^{m_1, n_1}\left[zt^{\lambda_1}(ct^{\mu}+d)^{\upsilon_1}\left|\begin{matrix}(a_j, \alpha_j)_{1,p_1}\\(b_j, \beta_j)_{1,q_1}\end{matrix}\right.\right] S_{n_2}^{m_2}\left[y\, t^{\lambda_2}(ct^{\mu}+d)^{\upsilon_2}\right]\right].$$

$$= \sum_{v=0}^{n} \sum_{u=0}^{v} \frac{(-v)_u}{v!\,u!}(-\alpha - pu)_n (-1)^n \beta^v e^{-n+pv} \sum_{h=0}^{[n_2/m_2]} \sum_{i=0}^{\infty}$$

$$\frac{(-n_2)_{m_2 h}}{g!} A_{n_2,h}\, y^h \frac{c^i d^{-i+\upsilon+\upsilon_2 h}}{i!}\, x^{\lambda+\lambda_2 h+\mu i-n+pv}\, \Gamma(\alpha+(-n+pv)^{\sigma}+1)$$

$$H^{m_1,n_1+2}_{p_1+2,\,q_1+2}\left[z\, x^{\lambda_1} d^{\upsilon_1} \left| \begin{matrix} (1-(\eta+\lambda+\rho r(-n+pv)+\lambda_2 h+\mu i+1)/r,\lambda_1/r) \\ (b_j,\beta_j)_{1,q_1},\ (i-\upsilon-\upsilon_2 h,\upsilon_1), \end{matrix} \right. \right.$$

$$\left. \begin{matrix} (-\upsilon-\upsilon_2 h,\upsilon_1),(a_j,\alpha_j)_{1,p_1}, \\ (-(\eta+\lambda+\lambda_2 h+\mu i+1)/r)-\alpha-(\rho+\sigma)(-n+pv),\lambda_1/r) \end{matrix} \right] \quad \dots (3.5)$$

where

$$R^*[f(x)] = r\, x^{-\eta-r\alpha-1} \int_0^x t^{\eta} (x^r-t^r)^{\alpha}\, H^r_n\left[e\left(\frac{t^r}{x^r}\right)^{\rho}\left(1-\frac{t^r}{x^r}\right)^{\sigma}\alpha,\beta\right] f(t)\, dt \quad (3.6)$$

is the *FIO* involving Gould and Hopper polynomial $H^r_n[z]$ as kernel [2].

Acknowledgement

The author is gratful to Dr. S.P. Goyal, University of Rajasthan, Jaipur for his able guidance and constant help during the preparation of this paper.

REFERENCES

[1] Rajni Agrawal, R.S. Pareek and M.Saigo, A general fractional integral formula, *J. Frac. Calculus,* *7* (1995), 55–60.

[2] H.W. Gould and A.T. Hopper, Operational formulas connected with two generalizations of Hermite polynomials, *Duke Math. J.* **29** (1962), 51–63.

[3] S.K.Raizada, A study of unified representation of special functions of mathematical physics and their use in statistical and boundary value problems, *Ph.D. thesis, Bundelkhand Univ., India,* 1991.

[4] M. Saigo, S.P. Goyal and S. Saxena, A theorem relating a generalized Weyl fractional integral, Laplace and Verma transform with Applications. (To appear).

[5] H.M. Srivastava, A contour integral involving Fox's *H*-function, Indian *J. Math.* **14** (1972), 1–6.

[6] H.M. Srivastava, K.C. Gupta and S.P. Goyal, *The H-functions of One and Two Variables with Applications,* South Asian Publishers, New Delhi, Madras, 1982.

[7] H.M. Srivastava and N.P. Singh, The integration of certain products of the multivariable *H*-function with a general class of polynomials, *Rend. Circ. Mat. Palermo (2)* **32** (1983), 157–187.