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(H,Hyp (R)) - HYPERRING

By

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ABSTRACT

The aim of this paper is to generalize the notion of *(H ,R)* – Hyperring introduced by De Salvo [2], a special attention is paid to the quotient hyperrring.

1. Introduction. A hypering $<R , + , \bullet>$ in the general sense is a non empty set R with two hyperoperations $+ : R \times R \rightarrow \rho(R)$, $\bullet : R\times R \rightarrow \rho(R)$ where $\rho(R)$ is the power set of R, such that $<R , +>$ is a hypergroup in the sense of Marty and $(\cdot)$ is an asociative hyperoperation, which is distributive with respect to (+) , i. e $x.(y+z) \subset x.y + x.z$, $(y + z).x \subset y.z + z.x$ for all $x ,y, z \in R$. For a detailed study of hypergroups, hperrings one is referred to [1],[3], [4] and [6]. In this work we generalize the algebraic structure *(H , R)* – Hyperring introduced by Salvo [2] where R is a ring. Here we introduce a new class of *(H ,R)* Hyperrings in which R is a hyperring in the general sense. Mainly we focus our attention on the quotients of *(H , Hyp R)* – Hyperring with respect to two-sided hyperideals. In this structure we prove that H is a two – sided hyperideal of *(H , Hyp R)* – Hyperring.

2. Recalls and definitions [5].

Definition 2.1. Consider the hyperring $<R , + , \bullet>$. We denote with γ^*_R[7] the transitive clousure of the relation γ defined as folloes :

$x \gamma_R y$ *iff* $\exists z_i \in R$ and I_k ,k finite index sets such that

$\{x ,y\} \subseteq \sum_{k \in K} (\prod_{i \in I_k} z_i) \in U_R$ where U_R is the set of all finite sums of product of elements of R and

$x \gamma^*_R y$ *iff* $\exists z_1, \dots , z_{n+1} \in R$ with $z_1 = x$, $z_{n+1} = y$ and $u_1 , \dots , u_{n+1}$ U_R such that $\{z_i , z_{i+1}\} \subseteq u_i$, for $i = 1, \dots , n$. γ^*_R is the minimal equivalence relation, for which the quotient R/γ^*_R is a ring. In the case that R/γ^*_R is a field, R is called a hyperfield in the general sense [7]. We denote by $\gamma^*_R (a)$ the equivalence class.

Definition 2.2. An element e is called an identity of a hypergroup $<H, o>$ iff $\forall y \in H$, $y \in eoy \cap yoe$. Also an element $x \in H$ is called an inverse of x iff there exists an identity $e \in H$ such that $e \in xox \cap xox$. In this paper we assume that there is a unique identity.

A sub-hypergroup T of $< R, +>$ is called sub-hyperring if $T.T \subseteq T$ and it is called left (respectively, right) hyperideal of R if $T.R \subseteq T$ (respectively, $R.T \subseteq T$) T is called left (respectively, right) reversible in R iff for all $(x,y) \in R^2$, the relation $y \in T+x$ (respectively, $y \in x+T$) implies that $x \in T+y$ (respectively, $x \in y+T$).

Theorem 3.2. [2]. Let δ be a two sided hyperideal of $< R, +, \bullet>$ such that δ is a left reversible sub-hypergroup of $<R,+>$. Then $<R/\delta, \blacksquare, \square>$ a hyperring where $(\delta+x)\blacksquare(\delta+y) = \{ \delta+ z:z \in \delta+x+\delta+y\}$ and $(\delta+x)\square(\delta+y) = \{ \delta+ w:w \in (\delta+x).(\delta+y)\} = \{\delta+w:w \in x.y\}$.

If $<A,+,\bullet>,<H,\oplus,o>$ are two hyperrings, then a hyperring homomorphism $f:<A,+,.> \to <H,\oplus,o>$ is a map such that $f(a+b) \subset f(a) \oplus f(b)$ and $f(a.b) \subset f(a)\, o\, f(b)$ for all $a,b \in A$.

3. Results.

Definiton : 3.1 Let $<H, *,o>$ be a hyperring such that $<H,*>$ is a hypergroup, and $<R, +, \bullet>$ be a hyperring, $\{A_i\}_{i \in R}$ be a family of non–empty sets indexed in R such that for all $i, j \in R$, $i \neq j$, $A_i \cap A_j = \phi$ and $A_o = H$ where o is the identity element of R $(o \in o+ i\ \forall i \in R)$ with respect to (+). We set $K = \bigcup_{i \in R} A_i$ and we define the hyperoperations $\oplus$ and Θ on K as follows :

$\forall x ,y \in H^2$, $x \oplus y = x^* y$, $x \odot y = xoy$, $\forall x,y \in A_i \times A_j \neq H^2$, $x \oplus y = A_{i+j}$ where A_{i+j} is defined by $A_{i+j} = \bigcup_{t \in i+j} A_t$ and $x \odot y = A_{i,j}$ where $A_{i,j} = \bigcup_{w \in i,j} A_w$.

Here we assume that o is a two sided absorbing element.

Theorem 3.2. $< K ,\oplus,\odot>$ is a hyperring if $<o>$ is associative.

Proof. $<K ,\oplus>$ is a hypergroup for

(i) Let $x,y,z \in K$, then one has : $(x \oplus y) \oplus z = (A_{i+j}) \oplus z = \bigcup_{t \in i+j} A_t \oplus z$ $= \bigcup_{w \in (i+j)+r} A_w = \bigcup_{w \in i+(j+r)} A_w = x \oplus (y \oplus z)$ where $x \in A_i$, $y \in A_j$ and $z \in A_r$. So $<K,\oplus >$ is a semihypergroup.

(ii) For all $x \in K$ one has, $x \oplus K = \bigcup_{k \in K} x \oplus k = \bigcup_{j \in R} A_{i+j} = \bigcup_{j \in R} A_t = \bigcup_{t \in i+R} A_t$ $= \bigcup_{t \in R+i} A_t = K \oplus x = \bigcup_{t \in R} A_t = K$.

This means that $< K,\oplus>$ is a quasi–hypergroup.

From (i), (ii) $< K ,\oplus,\odot>$ is a hypergroup.

(iii) ($\odot$) is an associative hyperoperation, the proof is similar to that of ⊕ except for the case, when $x \varepsilon H$, $z \varepsilon H$. In this case the reproduction axiom is necessary.

(iv) (⊙) is distributive with respect to ⊕ (It is clear).

Therefore $<K,\oplus,\odot>$ is a hyperring, which we call $(H, Hyp(R))$ – hyperring with support $K = \bigcup_{i \varepsilon R} A_i$.

Remark 3.3. $<K,\oplus,\odot>$ is commutative if $<R,+,\bullet>$ and $<H,*,o>$ are commutative.

Lemma 3.4. If $x \varepsilon A_i$ and $x \varepsilon A_{u+j}$ then $i \varepsilon u+j$ where $i,u,j \varepsilon R$.

Proof: $x \varepsilon A_{u+j}$ implies that $x \varepsilon \bigcup_{t \varepsilon u+j} A_t$, so there exists $r \varepsilon u+j$ such that $x \varepsilon A_r$, but $x \varepsilon A_i$. So $i = r$. Therefor $i \varepsilon u+j$.

Theorem 3.5: If $<K,\oplus,\odot>$ is a $(H, Hyp(R))$–hyperring with suport $k = \bigcup_{i \varepsilon R} A_i$ then $K/\gamma^*_k \cong R/\gamma^*_R$

Proof. Let $a \varepsilon K$, then there exists $r \varepsilon R$ such that $a \varepsilon A_r$, Let a_r denote a. If $x_t \varepsilon \gamma^*(a_r)$ where $x_t \varepsilon A_t, t \varepsilon R$. Then there exist $z_{r_1}, z_{r_2}, \ldots, z_{r_{n+1}} \varepsilon K$ such that $z_{r_1} = x_t$, $z_{r_{n+1}} = a_r$ and there exist $y_{t_i} \varepsilon K$ and M_i, I_{m_i}, $i = 1,2,\ldots, n$

finite index sets such that $\{z_{t_i}, z_{t_{i+1}}\} \subset \overset{\oplus}{\sum}_{m_i \varepsilon M} \left(\overset{\odot}{\prod}_{j \varepsilon I_{mi}} y_{t_j} \right)$, $i = 1,2, \ldots, n$.

The finite sum of products $\overset{\oplus}{\sum}_{m_i \varepsilon M} \left(\overset{\odot}{\prod}_{j \varepsilon I_{mi}} y_{t_j} \right) = A_{\overset{\oplus}{\sum}_{m_i \varepsilon M} \left(\overset{\odot}{\prod}_{j \varepsilon I_{mi}} y_{tj} \right)}$

where $y_{t_j} \varepsilon A_{t_j}$, $t_j \varepsilon R$, $j \varepsilon I_{mi}$.

From lemma 3.3 one has : $\{t_i, t_{i+1}\} \subseteq \sum_{m_i \varepsilon M} \left(\overset{\odot}{\prod}_{j \varepsilon I_{mi}} y_{t_j} \right)$ $i = 1, \ldots, n$. This means that : $r_1 \varepsilon \gamma^*_R (r_{n+1})$. But $r_1 = t$ and $r_{n+1} = r$. Therefore $t \varepsilon \gamma^*(r)$. Also the converse is true,

so there is an isomorphism $\sigma : K/\gamma^*_k \to R/\gamma^*_R : \gamma^*_k (a_r) \to \gamma^*_R (r)$,

Therefore $K/\gamma^*_k \cong R/\gamma^*_R$.

Corollary 3.6. If $<R,+,\bullet>$ is a ring then we have $K/\gamma^*_k \cong R$ which is theorem 3.3 [5]

Corollary 3.7. $= K/\gamma^*_k = \{A_i, i \varepsilon R/\gamma^*_R\} \cong R/\gamma^*_R$.

Definition 3.8. Let $<K,\oplus,\odot>$ be a $(H, Hyp(R))$–hyperring, $x' \varepsilon K$ is called an inverse of x in K iff $H \subseteq x \oplus x'$ and $H \subseteq x' \oplus x$.

Theorem 3.9. Let $<K,\oplus,\odot>$ be a $(H,Hyp(R))$–hyperring, $F \subset K$, $F \neq \Phi$, if $<R,+,\bullet>$ is a regular hyperring then F is a left (respectively, right) hyperideal of K iff $F = \bigcup_{i \varepsilon E} A_i$ where E is a left (respectively, right) regular hyperideal of R.

Proof. Since F is a left hyperideal of K the $H = F \odot H \subset F$. Let $x \varepsilon F-H$, then there exists $i \varepsilon R$, $i \neq o$ such that $x \varepsilon A_i$, it is implied that $A_i = x \oplus y \subset F$ where $y \varepsilon A_0 = H$. Since R is regular, then i has an inverse i' say. Let $z \varepsilon A_{i'}$, then $x \oplus z = A_{i+i'} = \bigcup_{t \varepsilon i+i'} A_i \supset H$. Consequently, from definition (3.8) z is an inverse of x in K and since F is a subhypergroup of K, then $z \varepsilon F$. Therefore $z \oplus y = A_i \subset F$. From the clousure of $\oplus$ in F we have $F = \bigcup_{i \varepsilon E} A_i$. Also if $i,j \varepsilon E$ then there exist $x \varepsilon A_i$ and $y \varepsilon A_j$ such that $x \oplus y = A_{i+j} \subseteq F$. Hence $i + j \subseteq E$, this means that $E + E \subseteq E$ and $i + E = E+i$ can be proved in a similar way, therefore E is a regular subhypergroup of R . It remains to prove that if $i \varepsilon E$ and $J \varepsilon R-\{0\}$, then $i.j \subseteq E$. Since $x \odot y = A_{i+j} \subset E$ then E is a left regular hyperideal of R.

Conversely. Lef $F = \bigcup_{i \varepsilon E} A_i$ where E is a regular hyperideal of R and consider $x \varepsilon F$, $k \varepsilon K$, then there exist $A_i, i \varepsilon E$ and A_r, $r \varepsilon R$ such that $x \varepsilon A_i$ and $k \varepsilon A$. Hence $x \odot k = A_{i,r}$ and since $i.r \subset E$, then $x \odot K = \bigcup_{t \varepsilon i-r} A_t = \bigcup_{t \varepsilon E} A_t$, it is implied that $x \odot k \subset F$.

Let i' be the inverse element of i in E, then $H \subset A_{i+i'} \subset F$ moreever $x' \varepsilon A_{i'}$, is the inverse of x in F for $x' \oplus x = A_{i+i'} \supset H$. Simply one can prove that $F \oplus F \subset F$ and $x \oplus F \subset F \ \forall \ x \varepsilon F'$ So F is a regular left hyperideal of k.

Corollary 3.10. If $<R,+,\circ>$ is a ring, we obtain Propostion 4.1 [5].

Lemma : 3.11 If $F = \bigcup_{i \varepsilon E} A_i$ is a hyperideal of K, then F is a two-sided reversible subhypergroup of R.

Proof. Let $a \varepsilon F \oplus b$ where $b \notin F$, and $b \varepsilon A_j$ for some $j \varepsilon R-E$ then $a \varepsilon \bigcup_{t \varepsilon E+j} A_t$, this impies that $a \varepsilon A_i$ for some $i \varepsilon E+j$ and since E is reversible in R, it is implied that $j \varepsilon E+i$, so $b \varepsilon \bigcup_{r \varepsilon E+i} A_r = F \oplus a$, this means that F is reverible subhypergroup fo k.

Theorem . 3.12 There is a hyperring homomorphism
$\Psi : < K/F, \blacksquare, \square > <R/E, \blacksquare, \square >$.

Proof. from theorem (3.2), $<K/F, \blacksquare, \square>$ and $<R/E, \blacksquare, \square>$ are hyperrings. For all $x, y \varepsilon K$, there exist $i, j \varepsilon R$ such that $x \varepsilon A_i$, $y \varepsilon A_j$ and $(F \oplus x) \blacksquare (F \oplus y) = \bigcup_{t \varepsilon E+i+E+j} A_t$ and $(F \oplus x)\ (F \oplus y) = \bigcup_{w \varepsilon E+i\ ,j} A_w = \bigcup_{w \varepsilon (E+i),(E+j)} A_w$

Define $\Psi : K/F \rightarrow R/E$ by $\Psi\ (F \oplus x) = \{E + r : r \varepsilon E+i\}$, we have

$\Psi\ [(F \oplus x) \blacksquare (F \oplus y)] = \{E + t : t \varepsilon E+i+E+j\} =$

$= (E+i) \blacksquare (E+j) \subset \Psi (F \oplus x) \blacksquare \Psi(F \oplus y)]$

Similarly, we can prove that $\Psi [(F \oplus x) \underline{-} \Psi(F \oplus y)] \subset \Psi (F \oplus x) \underline{-} \Psi (F \oplus y)]$

Consequently Ψ is a hyperring homomorphism.

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REFERENCES

[1] P. Corsini, Hypergroupes reguliers et hypermodules, *Ann. Univ. Ferrara Sez. VII* Sc. Mat. **20** (1975), 121-135.

[2] M. De Salvo, Iperanelli ed Ipercorpi, *Ann. Sci. Univ. Clermont II. Ser. Math.* **22** (1985), 89–107.

[3] M. Krasner, Approx. des corps value compl. decaract. $p = 0$ per ceux de caract. *0, Act Coll. Alg. Sup. C.B.R.M.* (1956), 19-22.

[4] J. Mittas, Sur les hyperanneaux et les hypercorps, *Mathematica Balkanica,* **3** (1937), 368-382.

[5] S. Spartlis, *(H,R)* - Hyperring *Fourth International Congress On Aha* (1990), 187-195.

[6] D. Stratigopoulos, Hyperanneaux non commutatifs, *C.R. Acad. Sc., Paris,* **269** (1969), 489-492.

[7] T. Vougiouklis, The fundamental relation in hyperrings. The general hyperfield. *Fourth International Congress On Aha* (1990), 203-211.