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CONTOUR INTEGRALS ASSOCIATED WITH TWO GENERAL CLASS OF POLYNOMIALS FOX'S *H*-FUNCTION AND THE MULTI-VARIABLE *H*-FUNCTION

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ABSTRACT

Here, we present two contour integrals based on a lemma, for the multivariable H-function [8] involving the product of Fox's H-function [1] and two general class of polynomials [6] with essentially arbitrary coefficients. By assigning suitable suitable values to the coefficients, the main result of two theorems can be reduced to integrals involving for instance the classical polynomials of Jacobi, Laguerre and Hermite. The multivariable H-function occuring in each of our main results can be reduced to the generalised Lauricella's hypergeometric functions of several complex variables.

1. INTRODUCTION AND NOTATIONS Srivastava and Panda [8] have introduced the multivariable H-function

$$H^{0,\,\lambda\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{A,\,C\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}\left(\begin{matrix}[(a):\theta';\ldots;\theta^{(r)}]: \\ [(c):\psi';\ldots;\psi^{(r)}]:\end{matrix}\;\begin{matrix}[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\; z_1,\ldots,z_r\right) \qquad \ldots(1.1)$$

The defining integral of the above function, its various special cases and other details can be found in the paper referred to above. For the sake of brevity

$$\Lambda_i = \sum_{j=1}^{\lambda}\theta_j^{(i)} + \sum_{j=1}^{B^{(i)}}\phi_j^{(i)} - \sum_{j=1}^{C}\psi_j^{(i)} - \sum_{j=1}^{D^{(i)}}\delta_j^{(i)} < 0 \qquad \ldots(1.2)$$

$$\Omega_i = -\sum_{j=\lambda+1}^{A}\theta_j^{(i)} + \sum_{j=1}^{v^{(i)}} - \sum_{j=1+v^{(i)}}^{B^{(i)}} - \sum_{j=1}^{C}\psi_j^{(i)}$$

$$+ \sum_{j=1}^{u^{(i)}}\delta_j^{(i)} - \sum_{j=1+u^{(i)}}^{D^{(i)}}\delta_j^{(i)} > 0,\ \forall\, i\ \varepsilon\ 1, \ldots, r. \qquad \ldots(1.3)$$

2. LEMMA If $h > 0$, $c > -Re(y)$, $Re(\rho) > 0$, m, m' are arbitrary positive integers and $A_{n,\alpha}$, $A_{n',\alpha'}$ are arbitrary constants, real or complex then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{hx}(x+y)^{-\rho} S_n^m(l(x+y)^{-1}) S_{n'}^{m'}(l'(x+y)^{-1}) \cdot H_{P,Q}^{M,N}\left[y'(x+y)^{-\sigma} \left| \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix} \right. \right] dt$$

$$= \sum_{g=1}^{M} \sum_{G=0}^{\infty} \sum_{\alpha}^{[n/m]} \sum_{\alpha'}^{[n'/m']} \frac{(-1)^G \phi(\eta_G)}{G!\, F_g} (y') \frac{\eta_G (-n)_{m\alpha} A_{n,\alpha}}{\alpha!} l^{\alpha} \frac{(-n')_{m'\alpha'}}{\alpha'!} A_{n',\alpha'} (l')^{\alpha'} e^{-hy} \frac{h^{\rho+\alpha+\alpha'+\sigma\eta_G-1}}{\Gamma(\rho+\alpha+\alpha'+\sigma\eta_G)}, \quad \text{...(2.1)}$$

given that

$$\sigma > 0, E_j > 0, j = 1, ..., \rho; F_j > 0, j = 1, ..., Q;$$

and $|\arg(y')| < \frac{T\pi}{2}, T > 0$

$$\left(T = \sum_{j=1}^{n} E_j - \sum_{j=n+1}^{P} E_j + \sum_{j=1}^{m} F_j - \sum_{j=m+1}^{Q} F_j \right).$$

Proof of the Lemma The assertion (1) of the Lemma follows at once by applying the definition of Fox's H-function and multivariable H-function in conjunction with the well-known Hankel's contour integral for the Gamma function which is given in the following form

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{hx}(x+y)^{-\sigma} dt = \frac{h^{\rho-1} e^{-hy}}{\Gamma(\rho)} \quad \text{...(2.2)}$$

$h > 0$, where for convergence $c > -Re(y)$ and $Re(\rho) > 0$.

3. THE MAIN CONTOUR INTEGRAL The main results of this paper are the contour integrals in the following theorem :

Theorem-1 With Λ_i and Ω_i defined by (1.2) and (1.3) respectively, let $\Lambda_i \leq 0$ and $|\arg(z_i)| < \Omega_i \pi/2, \forall i = 1, ..., r$ ...(3.1)

where the both equalities hold for suitably restricted values of the complex variables $z_1, ..., z_r$. Now, the function $H_{P,Q}^{M,N}$ be defined by [4] and [5], general class of polynomials be defined by [6] and lè $h > 0$ $c > -Re(y)$, then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{hx}(x+y)^{-\rho}\, S_n^m\,(l(x+y)^{-1})\, S_{n'}^{m'}\,(l'(x+y)^{-1})$$

$$H_{P,Q}^{M,N}\left[\, y'\,(x+y)^{-\sigma} \,\middle|\, \begin{matrix}(e_p, E_p)\\ (f_Q, F_Q)\end{matrix}\right].\, H(z_1(x+y)^{-\rho_1},\ldots, z_r(x+y)^{-\rho_r})\, dt$$

$$= \sum_{g=1}^{m}\sum_{G=0}^{\infty}\sum_{\alpha}^{[n/m]}\sum_{\alpha'}^{[n'/m']} \frac{(-1)^G\,\phi(\eta_G)}{G!\,F_g}\,(y')^{\eta_G}\,\frac{(-n)_{m\alpha}A_{n,\alpha}}{\alpha!}\,(l)^{\alpha}$$

$$\frac{(-n')_{m'\alpha'}}{\alpha'!}A_{n',\alpha'}\,(l')^{\alpha'}\,h^{\rho+\alpha+\alpha'+\sigma\eta_G-1}\,e^{-hy}$$

$$H\begin{matrix}0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})\\ A,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]\end{matrix}\left(\begin{matrix}[(a):\theta';\ldots;\theta^{(r)}]:\\ [(c):\psi';\ldots;\psi^{(r)}],\end{matrix}\right.$$

$$[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];$$

$$[1-\rho-\alpha-\alpha'-{}^{\sigma}\eta_G;\rho_1,\ldots,\rho_r]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];$$

$$\left. z_1h^{\rho_1},\ldots,z_rh^{\rho_r}\right) \quad \ldots(3.2)$$

provided that $\rho_j > 0$, $j = 1,\ldots, r$, $\sigma > 0$, $|\arg y'| < \frac{T\pi}{2}$ and $Re\,(\rho) > -c \sum_{j=1}^{r} \rho_j$.

Theorem-2 Under the hypothesis preceding the assertion (3) of the theorem-1

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{hx}(x+y)^{-\rho}\, H_{P,Q}^{M,N}\left[\, y'\,(x+y)^{-\sigma} \,\middle|\, \begin{matrix}(e_p, E_p)\\ (f_Q, F_Q)\end{matrix}\right]$$

$$S_n^m\,(l(x+y)^{-1})\, S_{n'}^{m'}\,(l'(x+y)^{-1})\, H(z_1(x+y)^{\rho_1},\ldots, z_r(x+y)^{\rho_r})\, dt$$

$$= \sum_{g=1}^{m}\sum_{G=0}^{\infty}\sum_{\alpha}^{[n/m]}\sum_{\alpha'}^{[n'/m']} \frac{(-1)^G\,\phi(\eta_G)}{G!\,F_g}\,(y')^{\eta_G}\,\frac{(-n)_{m\alpha}A_{n,\alpha}}{\alpha!}\,(l)^{\alpha}$$

$$\frac{(-n')_{m'\alpha'}}{\alpha'!}A_{n',\alpha'}\,(l')^{\alpha'}\,h^{\rho+\alpha+\alpha'+\sigma\eta_G-1}\,e^{-hy}$$

$$H\begin{matrix}0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})\\ A,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]\end{matrix}\left(\begin{matrix}[(a):\theta';\ldots;\theta^{(r)}]:\\ [(c):\psi';\ldots;\psi^{(r)}]:\end{matrix}\right.$$

$$[\rho+\alpha+\alpha'+\sigma\eta_G;\rho_1,\ldots,\rho_r]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];$$

$$[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];$$

$$\left. z_1h^{-\rho_1},\ldots,z_rh^{-\rho_r}\right) \quad \ldots(3.3)$$

provided that $\rho_j > 0$, $j = 1, ..., r$, $\sigma > 0$, $|\arg y'| < \frac{T\pi}{2}$ and $Re(\rho) > -c \sum_{j=1}^{r} \rho_j$.

Proofs of Theorem 1 and 2. The contour integrals formula of Theorem 1 can be proved by replacing the multivariable H-function in the integrand by its multiple contour integral [7]. Now we change the order of integration, evaluate the innermost integral by appealing to the assertion (2) of the lemma and then interpret the resulting multiple contour integral as an H-function of r-complex variables.

4. APPLICATIONS The contour integral formulas of Theorem 1 and 2 of the preceding section possesses manifold generality. The multivariable H-function may be transformed into G-functions, E-functions, Lauricella's functions, Hypergeometric functions, Legendre functions, Bessel functions, Hermite functions, Laguerre, functions and other functions in one, two or more arguments.

On other hand we can get several integral formulas for such special polynomials $\left(S_n^m [y] \right)_{n=0}^{\infty}$ by giving the values to n and n'.

Particular cases of Lemma (*I*) By applying our Lemma to the case of Hermite polynomials ([10], p.106, Eqn. (5.5.4) and [9]) by setting $S_n^2(x) \to x^{n/2} H_n\left(\frac{1}{2\sqrt{x}}\right)$ in which case $m = 2$, $A_{n,s} = (-1)^s$ and also letting $m' = 2$, $A_{n',s'} = (-1)^{s'}$, we obtain

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{hx} (x+y)^{-\rho} \cdot H_{P,Q}^{M,N}\left[y'(x+y)^{-\sigma} \,\middle|\, \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix} \right]$$
$$\left[(l(x+y)^{-1})^{n/2} H_n\left(\frac{1}{\sqrt{l(x+y)^{-1}}} \right) \right] \left[(l'(x+y)^{-1})^{n'/2} H_{n'}\left(\frac{1}{\sqrt{l'(x+y)^{-1}}} \right) \right] dt$$
$$= \sum_{g=1}^{m} \sum_{G=0}^{\infty} \sum_{s=0}^{[n/2]} \sum_{s'=0}^{[n'/2]} \frac{(-1)^G \phi(\eta_G)}{G!\, F_g} (y')^{\eta_G} \frac{(-n)_{2s}\, 2^n}{s!} \left(\frac{-l}{4} \right)^s$$
$$\frac{(-n')_{2s'}\, 2^{n'}}{s'!} \left(\frac{-l'}{4} \right)^{s'} \frac{e^{-hy}\, h^{\rho + \sigma\eta_G + s + s' - 1}}{\Gamma(\rho + {}^{\sigma}\eta_G + s + s')} \quad ...(4.1)$$

which holds under the same conditions as those required for (1).

(*II*) For the Laguerre polynomials ([10], p.101, Eqn.(5.1.6) and [9]) setting $S_n(x) \rightarrow L_n^{\alpha}(x)$ in which case $m = 1$ and $A_{n,s}\binom{n+\alpha}{n}\left(\frac{1}{(\alpha+1)_k}\right)$ and also taking $m' = 1$ and $A_{n',s'} = \binom{n'+\alpha'}{n'}\left(\frac{1}{(\alpha'+1)_{k'}}\right)$ we obtain the Lemma in the following form

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{hx}(x+y)^{-\rho} \,.\, H_{P,Q}^{M,N}\left[y'(x+y)^{-\sigma} \left| \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix} \right. \right]$$

$$\left[\sum_{s=0}^{n} \binom{n+\alpha}{n-s} \frac{(-l(x+y)^{-1})^s}{s!}\right]\left[\sum_{s'=0}^{n'} \binom{n'+\alpha'}{n'-s'} \frac{(-l'(x+y)^{-1})^{s}}{s'!}\right] dt$$

$$= \sum_{g=1}^{m} \sum_{G=0}^{\infty} \sum_{s=0}^{n} \sum_{s'=0}^{n'} \frac{(-1)^s \phi(\eta_G)}{G!\, F_g} (y')^{\eta_G} \frac{e^{-hy} h^{\rho+\sigma\eta_G+s+s'-1}}{\Gamma(\rho+\sigma\eta_G+s+s')}$$

$$\binom{n+\alpha}{n-s}\frac{(-x)^s}{s!}\binom{n'+\alpha'}{n'-s'}\frac{(-x')^{s'}}{s'!} \qquad \ldots(4.2)$$

which holds under the same conditions as those required for (1).

(*III*) For Lauricella's functions taking $\lambda = A$, $u^{(i)} = 1$, $v^{(i)} = B^{(i)}$, $D^{(i)} =$ $\forall\, i = 1, \ldots, r$, we obtain

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{hx}(x+y)^{-\rho} \,.\, H_{P,Q}^{M,N}\left[y'(x+y)^{-\sigma} \left| \begin{matrix} (e_p, E_p) \\ (f_Q, f_Q) \end{matrix} \right. \right]$$

$$S_n^m\,(l(x+y)^{-1})\, S_{n'}^{m'}\,(l'\,(x+y)^{-1})$$

$$F^{A\,:\,B'\,;\ldots;\,B^{(r)}}_{C\,:\,D'\,;\ldots;\,D^{(r)}}\left(\begin{matrix} [1-(a):\theta';\ldots;\theta^{(r)}]: \\ [1-(c):\psi';\ldots;\psi^{(r)}]: \end{matrix} \right.$$

$$\left. \begin{matrix} [1-(b'):\phi'];\ldots;[1-(b^{(r)}):\phi^{(r)}];\, z_1\,(x+y)^{-\rho_1} \\ \vdots \\ [1-(d'):\delta'];\ldots;[1-(d^{(r)}):\delta^{(r)}];\, z_2\,(x+y)^{-\rho_r} \end{matrix} \right) dt$$

$$= \sum_{g=1}^{m} \sum_{G=0}^{\infty} \sum_{\alpha=0}^{[n/m]} \sum_{\alpha'=0}^{[n'/m']} \frac{(-1)^G \phi(\eta_G)}{G!\, F_g} (y')^{\eta_G} h^{\sigma\eta_G+\alpha+\alpha'+\rho-1} e^{-hy}$$

$$\frac{(-n)_{m\alpha}}{\alpha!} A_{n,\alpha}\,(l)^{\alpha}\, \frac{(-n')_{m'\alpha'} A_{n',\alpha'}}{\alpha'\,!}\,(l')^{\alpha'}$$

$$. F^{A:B';...;B^{(r)}}_{C+1:D';...;D^{(r)}} \left(\begin{matrix} [1-(a):\theta';...;\theta^{(r)}]: \\ [1-(c):\psi';...;\psi^{(r)}], \end{matrix} \right.$$

$$[1-(b'):\phi'];...;[1-(b^{(r)}):\phi^{(r)}];$$

$$[1-\rho+\sigma\eta_G-\alpha-\alpha':\rho_1,...\rho_r]:[1-(d'):\delta'];...;[1-(d^{(r)}):\delta^{(r)}];$$

$$\left. z_1 h^{\rho_1};...;z_r h^{\rho_r} \right) \quad ...(4.3)$$

valid under the same conditions as given in the equation (4).

(*IV*) For Hermite polynomial in the theorem (1), we get

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{hx}(x+y)^{-\rho}\,.\,H^{M,N}_{P,Q}\left[y'(x+y)^{-\sigma} \middle| \begin{matrix}(e_p, E_p)\\(f_Q, F_Q)\end{matrix}\right]$$

$$\left[(l(x+y)^{-1})^{n/2} H_n\left(\frac{1}{\sqrt{l(x+y)^{-1}}}\right)\right]\left[(l'(x+y)^{-1})^{n'/2} H_{n'}\left(\frac{1}{\sqrt{l'(x+y)^{-1}}}\right)\right] dt$$

$$. H(z_1(x+y)^{-\rho_1},... z_r(x+y)^{-\rho_r}]\,dt$$

$$= \sum_{g=1}^{M}\sum_{G=0}^{\infty}\sum_{s=0}^{[n/2]}\sum_{s'=0}^{[n'/2']} \frac{(-1)_G\,\phi(\eta_G)}{G!\,F_g}(y')^{\eta_G} h^{\sigma\eta_G+s+s'+\rho-1} e^{-hy}$$

$$\frac{(-n)_{2s}\,2^n}{s!}\left(\frac{-l}{4}\right)^s \frac{(-n')_{2s'}\,2^{n'}}{s'!}\left(\frac{-l'}{4}\right)^{s'}$$

$$H^{0,\lambda:(u',v');...;(u^{(r)},v^{(r)})}_{A,C+1:[B',D'];...;[B^{(r)},D^{(r)}]} \left(\begin{matrix} [(a):\theta';...;\theta^{(r)}]: \\ [(c):\psi':...;\psi^{(r)}]: \end{matrix} \right.$$

$$[(b'):\phi'];...;[(b^{(r)}):\phi^{(r)}];$$

$$[1-\rho-\sigma\eta_G-s-s';\rho_1,...,\rho_r]:[(d'):\delta'];...;[(-d^{(r)}):\delta^{(r)}];$$

$$\left. \begin{matrix} z_1 h^{\rho_1} \\ \vdots \\ z_2 h^{\rho_2} \end{matrix} \right) \quad ...(4.4)$$

which holds under the same conditions as given in equation (4).

(*V*) Also by applying our theorem 1 to the case of Laguerre polynomials, we obtain

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{hx}(x+y)^{-\rho}\,.\,H^{M,N}_{P,Q}\left[y'(x+y)^{-\sigma} \middle| \begin{matrix}(e_p, E_p)\\(f_Q, F_Q)\end{matrix}\right]$$

$$L_n^{(\alpha)}(l(x+y)^{-1})\, L_{n'}^{(\alpha')}(l'(x+y)^{-1}).H(z_1(x+y)^{-\rho_1},\dots,z_r(x+y)^{-\rho_r}]dt$$

$$= \sum_{g=1}^{m}\sum_{G=0}^{\infty}\sum_{s=0}^{[n]}\sum_{s'=0}^{[n']} \frac{(-1)^G\,\phi(\eta_G)}{G!\,F_g}(y')^{\eta_G}\, h^{\sigma\eta_G+s+s'-1}\, e^{-hy}\, dt$$

$$\frac{(-l)^s}{s!}\binom{n+\alpha}{n-s}\frac{(-l')^{s'}}{s'!}\binom{n'+\alpha'}{n'-s'}$$

$$H^{\,0,\lambda:(u',v');\dots;(u^{(r)},v^{(r)})}_{\,A,C+1:[B',D'];\dots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}[(a):\theta';\dots;\theta^{(r)}]:\\ [(c):\psi';\dots;\psi^{(r)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\dots;[(b^{(r)}):\phi^{(r)}];\\ [1-\rho+\sigma\eta_G-s-s';\rho_1,\dots,\rho_1]:[(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\begin{matrix}z_1h^{\rho_1}\\ \vdots\\ z_2h^{\rho_r}\end{matrix}\right) \quad \dots(4.5)$$

valid for the same conditions as given in equation (4).

(*VI*) Letting $n' \to 0$, the above five results of particular cases reduce to the known results obtained by Chaurasia and Tyagi [3].

(*VII*) Letting $n' \to 0$, $n \to 0$ the above five results of particular cases reduce to the known results obtained by V.B.L. Chaurasia [2].

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