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EXISTENCE THEOREMS FOR COMMON SOLUTIONS OF DIFFERENTIAL EQUATIONS IN BANACH SPACE

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(The paper is dedicated in the loving memory of the late beloved mother Smt. Kashubai Chandrabhan Dhage who expired on 17-10-1994)

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ABSTRACT

The aim of the present paper is to establish the stability of the common solution of two differential equations in Banach spaces via fixed point technique.

1. INTRODUCTION. At present there is an abundance of literature available on the stability of differential equations with the different notions and with the different approaches. See for example. Hirsch and Smale [2], Lakshmikantham and Leela [3] and Lasalle [4]. The stability of differential equations via the fixed point technique is well-known in the literature. Recently the present author proved a common fixed point theorem for a pair of mappings in Banach spaces and which has been applied to prove the common solution of two differential equations in Banach spaces [1]. In this paper the simultaneous stability of two differential equations in Banach spaces is established. Before proving the main resutls of this paper, we first prove a fixed point theorem which will be used in the sequel. We need the following preliminaries.

Let E denote a Banach space. A subset K of E is said to be a cone in E if (i) $k_r + k \leq K$ (ii) $\lambda k \subseteq K$, whenever $\lambda \geq 0$, and (iii) $K \cap -K = \{0\}$. The relation $x \leq y$ if and only if $y - x \in K$, defines a partial ordering in E.

Definition 1.1 : Two mappings A and B on a Banach space into itself with a order relation $\leq$ induced by the closed cone K in X, are said to be **isotone increasing** if $A(x) \leq B(A(x))$ and $B(x) \leq A(B(x))$ for all $x \in X$. If the reverse inequalities are satisfied, then A and B are called weakly **isotone decreasing** on X. Finally, A and B are called **weakly isotone** on X if they are either **weakly isotone** increasing or decreasing on X.

Theorem 1.1 : Let S be a non-empty, closed, convex and bounded subset of a Banach space X with an order relation $\leq$ induced by the closed cone K in X and let A, $B : S \to S$ be two completely continuous and weakly isotone mappings. Then A and B have a common fixed point.

Proof : Let S be arbitrary and consider the sequence $\{x_n\}$ in S defined by

$$x_0 = x,\ x_{2n+1} = Ax_{2n},\ x_{2n+2} = BX_{2n+1} \qquad \ldots(1.1)$$

for $n = 0, 1, 2, \ldots$.

Suppose that A and B are weakly isotone increasing on S. Then by nature of the sequence $\{x_n\}$, we get

$$x_1 \le x_2 \le \ldots \le x_n \le \ldots \qquad \ldots(1.2)$$

Since A and B are completely continuous $A(S)$ and $B(S)$ are relatively compact and hence $\overline{A(S)}$ and $\overline{B(S)}$ are compact. As $\{x_n\} \subseteq \{x_0\} \cup \overline{A(S)} \cup \overline{B(S)}$ and $\{x_n\}$ is monotone increasing, it has a unique limit point x^* in S such that $\lim_{n\to\infty} x_n = x^*$, and every subsequence of $\{x_n\}$ converges to the same limit point. By continuity of A and B, we obtain

$$x^* = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} A(x_{2n}) = A\left(\lim_{n\to\infty} x_{2n}\right) = Ax^*$$

$$x^* = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} B(x_{2n+1}) = B\left(\lim_{n\to\infty} x_{2n+1}\right) = Bx^*$$

Thus x^* is a common fixed point of A and B. Similarly, if A and B are weakly isotone decreasing on S, then it can be shown that the sequence $\{x_n\}$ in S defined by (1.1) is monotone decreasing and converges to the unique limit point which is again a common fixed point of A and B. This completes the proof.

Corollary 1.1 : Let S be a non-empty, closed, convex and bounded subset of a Banach space X with an order relation $\le$ induced by the closed cone K in X and let $A, B : S \to S$ be two weakly isotone mappings. If A is continuous and B is a completely continuous mapping on S, then A and B have a common fixed point.

Proof : Let $x \in S$ be arbitrary and consider a sequence $\{x_n\}$ in S defined by (1.1).

Now suppose that A and B are weakly isotone increasing on S. Then by nature of the sequence, we get

$$x_1 \le x_2 \le \ldots \le x_n \le \ldots \qquad \ldots(1.3)$$

Since A is continuous and B is completely continuous, AoB is completely continuous on S. Therefore it follows that $AB(S)$ is relatively compact and hence $\overline{AB(S)}$ and $\overline{B(S)}$ are compact in X. As $\{x_n\} \subseteq \{x_0, x_1\} \cup \overline{B(S)} \cup \overline{AB(S)}$, in view of (1.3) it follows that the sequence $\{x_n\}$ converges to a unique limit point. The rest of the proof is similar to Theorem 1.1. The proof is complete.

Corollary 1.2 : Let S be a non-empty, closed, convex and bounded subset of a Banach space X with an order relation $\leq$ induced by the closed cone K on X and let $A, B : S \to S$ be two weakly isotone mappings. Further, if A is a contraction and B is completcly continuous, then A and B have a unique common fixed point.

Proof : Since every contraction mapping is continuous, A is continuous on S. By corollary 1.2, the mapping A and B have a common fixed point. But by the Banach contraction mapping principle, it follows that A has a unique fixed point. So A and B have a unique common fixed point. This completes the proof.

In the following section, we prove the simultaneous stability of two differential equations, particularly initial value problems in Banach spaces, by an applications of the results proved in this section.

2. STABILITY OF DIFFERENTIAL EQUATIONS

Let R denote the real line and $J = [t_0, t_0 + a] \subset R$, for some t_0, $a \in R$, $a > 0$ a closed and bounded interval. Let E denote a Banach space with norm $\|.\|_E$ and an order relation $\leq$ induced by a closed cone K in it. Now consider the nonlinear differential equations (in short DEs) with the same initial condition (for convenience),

$$\left.\begin{aligned} x' &= f(t, x) \\ x(t_0) &= x_0 \in E \end{aligned}\right\} \qquad \text{...(2.1)}$$

and

$$\left.\begin{aligned} x' &= g(t, x) \\ x(t_0) &= x_0 \in E \end{aligned}\right\} \qquad \text{...(2.2)}$$

for $t \in J$, where $f, g \in C(J \times E, E)$.

We seek the simultaneous stability of the common solution of the DEs (1.1) and (1.2) in the space $X = C(J, E)$ of all continuous E-valued functions on J. Define a norm $\|.\|$ in X by

$$\| X \| = \sup_{t \in J} \| x(t) \|_E \qquad \text{...(2.3)}$$

Clearly X is a Banach space with this supremum norm. We introduce an order relation $\leq$ in X as follows. Let K denotes a cone in X defined by

$$K = \{x \in X \mid x(t) \in K, \text{ for all } t \in J\} \qquad \text{...(2.4)}$$

Let $x, y \in X$, then by $x \leq y$ we mean $y - x \in K$, or equivalently $y(t) - x(t) \in K$ for all $t \in J$, i.e. $x(t) \leq y(t)$ for all $t \in J$. Obviously K is a closed cone in X.

Definition 2.1 : The differential equations (2.1) and (2.2) are said to have a stable common solution P if for $\varepsilon > 0$, there exists an $n > 0$ such that $\|x_0\| \leq n$ implies $\| p(t) \|_E \leq \varepsilon$ for all $t \in J$.

We consider the following set of assumptions :

(A_1) For given $\varepsilon > 0$ there exists a $\delta > 0$, $a\delta > 1$, such that

$$\| f(t,x) - f(t,y) \|_E \leq \delta \| x - y \|_E$$

and
$$\| g(t,x) - g(t,y) \|_E \leq \delta \| x - y \|_E$$

for all $(t, x), (t, y) \in J \times E$ whenever $\| x \|_E \leq \varepsilon$, $\| y \|_E \leq \varepsilon$.

(A_2) $f(t, \theta) = \theta = g(t, \theta)$ for all $t \in J$, where θ is the zero vector of E.

(A_3)
$$f(t, x(t)) \leq g\left(t, \int_0^t f(s, x(s))\, ds + x_0 \right)$$

and
$$g(t, x(t)) \leq f\left(t, \int_{t_0}^t f(s, x(s))\, ds + x_0 \right)$$

for $(t, x) \in J \times X$ and for a fixed element $x_0 \in E$ gien in (2.1) and (2.2).

(A_4) $f(t, x)$ and $g(t, x)$ are continuous in t, uniformly for $x \in E$ with $\| x \|_E < \infty$.

(A_5) The function $f(t, x)$ satisfies the Lipschitz condition in x, uniformly for $t \in J$, i.e. there exists a constant $L > 0$ such that

$$\| f(t,x) - f(t,y) \|_E \leq L \| x - y \|_E$$

for all $(t, x), (t, y) \in J \times E$.

Theorem 2.1 : Assume $(A_1) - (A_4)$. Then the *DEs* (2.1) and (2.2) have a stable common solution on J, whenever $\| x_0 \|_E \leq b\,\varepsilon$ for some $b \in (0, 1)$.

Proof : Let $\epsilon > 0$ be given and take $\delta = \dfrac{1-b}{a}$ for some $b \in (0, 1)$, which is possible in view of $a\delta < 1$. Then by (A_1) and (A_2), we have

$$\left.\begin{aligned} \| f(t,x) \|_E &\leq \frac{1-b}{a}\varepsilon \\ \text{and} \quad \| g(t,x) \|_E &\leq \frac{1-b}{a}\varepsilon \end{aligned}\right\} \qquad \text{...(2.5)}$$

for all $(t, x) \in J \times E$.

Define a subset $\overline{S}(\varepsilon)$ of the Banach space X by

$$\overline{S}(\varepsilon) = \{x \in X / \| x \| \leq \varepsilon\} \qquad \text{...(2.6)}$$

Clearly a subset $\overline{S}(\varepsilon)$ is a closed, convex and bounded subset of the Banach space X. Now the *DEs* (2.1) and (2.2) are equivalent to the integral equations,

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s))\, ds\ , \quad t \in J \qquad \text{...(2.7)}$$

and $$x(t) = x_0 + \int_{t_0}^{t} g(s, x(s))\, ds\,, \quad t \in J \qquad \ldots(2.8)$$

respectively.

Define two operators A and B on $\overline{S}(\varepsilon)$ by right hand sides of the equations (2.7) and (2.8), respectively. Then the problem of a common solution of the *DEs* (2.1) and (2.2) is reduced to finding the common fixed points of the operators A and B. We show that the operators A and B map $\overline{S}(\varepsilon)$ into itself. Let $x \in \overline{S}(\varepsilon)$ be arbitrary then by $(A_1) - (A_2)$, we obtain

$$\| Ax(t) \|_E \le \| x_0 \|_E + \int_{t_0}^{t} \| f(s, x(s)) \|_E \, ds$$

$$\le b\varepsilon + \frac{1-b}{a}\varepsilon . a = \varepsilon$$

for all $t \in J$ and hence A maps $\overline{S}(\varepsilon)$ into itelf. Similarly B also maps $\overline{S}(\varepsilon)$ into itself. Also the operators A and B are completely continuous on $\overline{S}(\varepsilon)$ in view of the hypotheses $(A_1) - (A_2)$ and (A_4). Next we show that the operators A and B are weakly isotone increasing on $\overline{S}(\varepsilon)$, then by (A_3)

$$Ax(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds \le x_0 + \int_{t_0}^{t} g\left(s, \int_{t_0}^{s} f(\tau, x(\tau))\, d\tau + x_0 \right) ds$$

$$= x_0 + \int_{t_0}^{t} g(s, Ax(s))\, ds$$

$$= B(Ax(t))$$

for all $t \in J$, and so $Ax \le B(Ax)$. Similarly $Bx \le A(Bx)$ for all $x \in \overline{S}(\varepsilon)$. This shows that A and B are weakly isotone increasing on $\overline{S}(\varepsilon)$. Now an application of Theorem 1.1 yields that the operators A and B have a common fixed point in $\overline{S}(\varepsilon)$. Consequently the *DEs* (2.1) and (2.2) have a common solution in $\overline{S}(\varepsilon)$. Thus there exists a common solution p of the *DEs* (2.1) and (2.2) such that $\| p(t) \|_E \le \varepsilon$ for all $t \in J$, whenever $\| x_0 \|_E \le b\varepsilon$ for some $b \in (0, 1)$. This, in view of defintion 2.1, implies that the *DEs* (2.1) and (2.2) have a stable common solution on J. This completes the proof.

Theorem 2.2 : Assume $(A_1) - (A_5)$. Further if $La < 1$ then the *DEs* (2.1) and (2.2) have a unique stable solution on J whenver $\| x_0 \|_E \le b\varepsilon$ for some $b \in (0, 1)$.

Proof : By Theorem 2.1, the *DEs* (2.1) and (2.2) have a stable solution on J. To prove the uniqueness, let u and v be two distinct stable common solutions of the *DEs* (2.1) and (2.2) on J. Then u and v satisfy the *DEs* (2.1) and (2.2) on J. Therefore, we have

$$u(t) = x_0 + \int_{t_0}^{t} f(s, u(s))\, ds, t \in J$$

and
$$v(t) = x_0 + \int_{t_0}^{t} f(s, v(s))\, ds\ , t\in J$$

Then

$$\| u(t) - v(t) \|_E \leq \int_{t_0}^{t} \| f(s, u(s)) - f(s, v(s)) \|_E \, ds$$
$$\leq La \| u(t) - v(t) \|_E$$

or
$$\| u - v \| \leq La \| u - v \|,$$

which is not possible since $La < 1$. Hence $u(t) = v(t)$ for all $t \in J$. This completes the proof.

Remark 2.1 : The assumption (A_3) in Theorems 2.1 and 2.2 can be replaced by the following assumptions $(A_3)'$.

$(A_3)'$ (i) $f(t, x)$ and $g(t, x)$ are non-decreasing in $x \in E$ uniformly for $t \in J$,

(ii) $f(t, x) \leq g(t, f(t, x))$ and $g(t, x) \leq f(t, g(t, x))$ for all $(t, x) \in J \times E$

$$(iii) \quad f(t, x(t)) \leq x_0 + \int_{t_0}^{t} f(s, x(s))\, ds$$

and
$$g(t, x(t) \leq x_0 + \int_{t_0}^{t} g(s, x(s))\, ds$$

for all $(t, x) \in J \times X$ and for a fixed element $x_0 \in E$ given in (2.1) and (2.2).

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