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A SOLUTION OF THE PARTIAL DIFFERENTIAL EQUATION OF HEAT CONDUCTION IN A ROD UNDER THE ROBIN CONDITION

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ABSTRACT

The role of special functions and their applications is becoming increasingly of much significance in technical applications. Space research and nuclear reactor also give rise to several problems of application of special functions.

This paper has been designed to discuss the application of certain products involving the multivariable H-function and two general class of polynomials in establishing a solution of the partial differential equation

$$\frac{\partial \theta}{\partial t} = \mu \frac{\partial^2 \theta}{\partial x^2}$$

related to a problem of heat conduction in a rod under the Robin condition. The result gives a number of known particular cases on specializing the parameters and may prove to be useful in several interesting situations appearing in the literature on mathemaitcal analysis, applied mathematics and mathematical physics.

1. INTRODUCTION. As an example of the application of the multivariable H-function in applied mathematics, the problem of obtaining a solution of a problem of flow of heat in a uniform rod under third kind of boundary conditions, i.e., the problem of heat conduction in a uniform rod under the Robin condition (at zero temperature with radiation at the ends inside the medium) will be consider in this problem. If the thermal coefficient is constant and their is no source of thermal energy, then a function $u\,(x, t)$ representing the temperature in a rod $(0 \le x \le L)$ must satisfy the heat equation

$$\frac{\partial \theta}{\partial t} = \mu \frac{\partial^2 \theta}{\partial x^2} \qquad ...(1.1)$$

and the initial conditions

$$\theta\,(x, 0) = f(x) \qquad ...(1.2)$$

$$\frac{\partial \theta}{\partial x}\,(0, t) - h\theta\,(0, t) = 0 \qquad ...(1.3)$$

$$\frac{\partial \theta}{\partial x}\,(L, t) + h\theta\,(L, t) = 0;\ h > 0 \qquad ...(1.4)$$

Here we shall consider

$$f(x) = \left(\sin \frac{\pi x}{L}\right)^{\sigma - 1} S_n^m\left[y\left(\sin \frac{\pi x}{L}\right)^{2\rho}\right] S_{n'}^{m'}\left[y'\left(\sin \frac{\pi x}{L}\right)^{2\rho'}\right]$$

$$H^{0,\,\lambda\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{A,\,C\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}\left(\begin{matrix} z_1\,(\sin \frac{\pi x}{L})^{2k_1} \\ \vdots \\ z_r\,(\sin \frac{\pi x}{L})^{2k_r} \end{matrix}\right) \qquad ...(1.5)$$

(For the multivariable H-function and a general class of polynomials, see [7] and [6], respectively).

The expression

$$e^{-\mu E_q^2 t}\,(m \cos E_q x + n \sin E_q x) \qquad ...(1.6)$$

satisfies the equation (1.1), Equation (1.6) also satisfies (1.3) and (1.4) on the condition that

$$E_q n - hm = 0 \qquad ...(1.7)$$

and $E_q(n \cos E_q L - m \sin E_q L) + h(m \cos E_q L + n \sin E_q L) = 0$ (1.8)

From (1.6) and (1.7), we get

$$A/B = Eq/h \qquad ...(1.9)$$

and

$$\tan E_q L = 2E_q h/(E_q^2 - h^2) \qquad ...(1.10)$$

where E_q is the qth positive root of (1.14). Then the solution of our problem takes the following elegant from

$$\theta\,(x, t) = \sum_{q=1}^{\infty} R_q \left(\cos E_q x + \frac{h}{E_q} \sin E_q x\right) e^{-\mu E_q^2 t} \qquad ...(1.11)$$

The following results will be required :

$$\int_0^L (\sin \frac{\pi x}{L})^{\sigma - 1} \sin \frac{E_p \pi x}{L} dx$$

$$= L.2^{1-\sigma} \sin \frac{E_p \pi}{2} \frac{\Gamma(\sigma)}{\Gamma(\frac{\sigma \pm E_p + 1}{2})}, R(\ ) > 0 \quad ...(1.12)$$

$$\int_0^l (\sin \frac{\pi x}{L})^{\sigma - 1} \cos \frac{E_p \pi x}{L} dx$$

$$= L.2^{1-\sigma} \cos \frac{E_p \pi}{2} \frac{\Gamma(\sigma)}{\Gamma(\frac{\sigma \pm E_p + 1}{2})}, R(\sigma) > 0 \quad ...(1.13)$$

$$\int_0^L (\cos E_q x + \frac{h}{E_q} \sin E_q x)(\cos E_p x + h_l E_p \sin E_p x)\, dx$$

$$= \begin{cases} 2(E_q^2)^{-1} [(E_q^2 + h^2)L + 2h], & p = q \\ 0, & p \neq q \end{cases} \quad ...(1.14)$$

where E_q is positive root of the non-algebraic equation

$$\tan EL = \frac{2hE}{(E^2 - h^2)} \quad ...(1.15)$$

We shall require the following definition (cf. Srivastava [6], p.1, eqn.(1.1)]

$$S_n^m [x] = \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} F_{n,s}\, x^s,\ n = 0, 1, 2, \ldots \quad ...(1.16)$$

where m is arbitrary positive integer and the coefficients $F_{n,s}$ $(n, s \geq 0)$ are arbitrary constants, real or complex. By suitably specializing the coefficients $F_{n,s}$ the polynomials $S_n^m[x]$ can be reduced to the well-known classical orthogonal polynomials such as Jacobi, Hermite, Legendre, Tchebycheff and Laguerre polynomials, etc.

2. THE MAIN INTEGRAL FORMULAE

$$\int_0^L (\sin \frac{\pi x}{L})^{\sigma - 1} \sin \frac{E_p \pi x}{L} S_n^m \left[y (\sin \frac{\pi x}{L})^{2\rho} \right]$$

$$S_{n'}^{m'}\left[y' (\sin \frac{x\pi}{L})^{2\rho'} \right] H^{0, \lambda : (u', v') ;...; (u^{(r)}, v^{(r)})}_{A, C : [B', D'] ;...; [B^{(r)}, D^{(r)}]} \begin{pmatrix} z_1 (\sin \frac{\pi x}{L})^{2k_1} \\ \vdots \\ z_r (\sin \frac{\pi x}{L})^{2k_r} \end{pmatrix} dx$$

$$= L.2^{1 - \sigma - 2\rho s - 2\rho' s'} \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} F_{n,s}\, y^s$$

$$\sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!} F_{n',s'}\, y'^{s'} \sin \frac{E_p \pi}{2}$$

$$H^{0, \lambda + 1 : (u', v') ;...; (u^{(r)}, v^{(r)})}_{A+1, C+2 : [B', D'] ;...; [B^{(r)}, D^{(r)}]}$$

$$\left(\begin{matrix} [1 - \sigma - 2\rho s - 2\rho' s' : 2k_1 ,..., 2k_r], [(a) : \theta' ,..., \theta^{(r)}] : \\ [(c) : \psi' ,..., \psi^{(r)}], [\frac{1 - \sigma - 2\rho s - 2\rho' s' \pm E_p}{2} : k_1 ,..., k_r] : \end{matrix} \right.$$

$$\left. \begin{matrix} [(b') : \phi'] ;...; [(b^{(r)}) : \phi^{(r)}] ; z_1\, 4^{-k_1} \\ \vdots \\ [(d') : \delta'] ;...; [(d^{(r)}) : \delta^{(r)}] ; z_r\, 4^{-k_r} \end{matrix} \right) \quad ...(2.1)$$

where $\rho > 0$, $\rho' > 0$, $k_i > 0$, $\text{Re}\,(\sigma + \sum_{i=1}^{r} k_i d_j^{(i)}/\delta_j^{(i)}) > 0$, $1 \le i \le r$, $1 \le j \le u^{(i)}$, $|\arg (z_i)| < T_i\, \pi/2$, $T_i > 0$, m, m' are arbitrary positive integers and the coefficients $F_{n,s}\,(n, s \ge 0)$ and $F_{n',s'}\,(n', s' \ge 0)$ are arbitrary constants, real or complex.

$$\int_0^L (\sin \frac{\pi x}{L})^{\sigma - 1} \cos \frac{E_p\, \pi x}{L} S_n^m \left[y (\sin \frac{\pi x}{L})^{2\rho} \right]$$

$$S_{n'}^{m'}\left[y' (\sin \frac{\pi x}{L})^{2\rho'} \right] H^{0, \lambda : (u', v') ;...; (u^{(r)}, v^{(r)})}_{A, C : [B', D'] ;...; [B^{(r)}, D^{(r)}]} \begin{pmatrix} z_1 (\sin \frac{\pi x}{l})^{2k_1} \\ \vdots \\ z_r (\sin \frac{\pi x}{l})^{2k_r} \end{pmatrix} dx$$

$$= L.2^{1 - \sigma - 2\rho s - 2\rho' s'} \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} F_{n,s}\, y^s$$

$$\sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!} F_{n',s'}\, y'^{s'} \cos \frac{E_p \pi}{2}$$

$$H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+2:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}[1-\sigma-2\rho s-2\rho' s':2k_1,\ldots,2k_r],[(a):\theta',\ldots,\theta^{(r)}]:\\ [(c):\psi',\ldots,\psi^{(r)}],\left[\dfrac{1-\sigma-2\rho s-2\rho' s'\pm E_p}{2}:k_1,\ldots,k_r\right]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];\; z_1\,4^{-k_1}\\ \vdots\\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\; z_r\,4^{-k_r}\end{matrix}\right) \qquad \ldots(2.2)$$

where $\rho > 0$, $\rho' > 0$, $k_i > 0$, $\mathrm{Re}\,(\sigma + \sum_{i=1}^{r} k_i\, d_j^{(r)}/d_j^{(i)}) > 0$, $1 \le i \le r$, $1 \le j \le u^{(i)}$, $|\arg(z_i)| < T_i\,\pi/2$, $T_i > 0$, m, m' are arbitrary positive integers and the coefficients $F_{n,s}$ $(n, s \ge 0)$ and $F_{n',s'}$ $(n', s' \ge 0)$ are arbitrary constants, real or complex. The integral formulae in (2.1) and (2.2) can be derived by making use of the definition of the multivariable H-function (Srivastava and Panda [7] and Srivastava [8], p.251) and of the results in (1.12), (1.13) and (1.16) respectively.

3. SOLUTION OF THE PROBLEM The solution of the problem to be established

$$\theta(x,t) = \frac{2L}{2^{\sigma+2\rho s+2\rho' s'-1}} \sum_{q=1}^{\infty} \sum_{s=0}^{[n/m]} \sum_{s'=0}^{[n'/m']} \frac{(-n)_{ms}}{s!}$$

$$F_{n,s}\, y^s\, \frac{(-n')_{m's'}}{s'!}\, F_{n',s'}\, y^{s'}\, \frac{E_q^2}{[(E_q^2+h^2)\,L+2h]}\, e^{-\mu E_q^2 t}$$

$$(\cos E_q x + \frac{h}{E_q}\sin E_q x)\,(\cos\frac{E_q\pi}{2} + \frac{h}{E_q}\sin\frac{E_q\pi}{2})$$

$$H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+2:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}[1-\sigma-2\rho s-2\rho' s':2k_1,\ldots,2k_r],[(a):\theta',\ldots,\theta^{(r)}]:\\ [(c):\psi',\ldots,\psi^{(r)}],[\dfrac{1-\sigma-2\rho s-2\rho' s'\pm E_q}{2}:k_1,\ldots,k_r]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];\; z_1\,4^{-k_1}\\ \vdots\\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\; z_r\,4^{-k_r}\end{matrix}\right). \qquad \ldots(3.1)$$

where $\rho > 0$, $\rho' > 0$, $k_i > 0$, $\text{Re}\,(\sigma + \sum_{i=1}^{r} k_i d_j^{(i)}/\delta_j^{(i)}) > 0$, $1 \le i \le r$, $1 \le j \le u^{(i)}$, $|arg\,(z_i)| < T_i\,\pi/2$, $T_i > 0$, m, m' are arbitrary positive integers and the coefficients $F_{n,s}\,(n, s \ge 0)$ and $F_{n',s'}\,(n', s' > 0)$ are arbitrary constants, real or complex. Derivation of (3.1).

If $t = 0$ then from (1.5) and (1.11), we have

$$(\sin \frac{\pi x}{L})^{\sigma - 1} S_n^m \left[y\,(\sin \frac{\pi x}{L})^{2\rho} \right] S_{n'}^{m'} \left[y'\,(\sin \frac{\pi x}{L})^{2\rho'} \right]$$

$$H^{0,\lambda : (u',v');...;(u^{(r)},v^{(r)})}_{A, C : [B',D'];...;[B^{(r)},D^{(r)}]} \left(\begin{matrix} z_1\,(\sin \frac{\pi x}{L})^{2k_1} \\ \vdots \\ z_r\,(\sin \frac{\pi x}{L})^{2k_r} \end{matrix} \right)$$

$$= \sum_{q=1}^{\infty} R_q\,(\cos E_q x + \frac{h}{E_q} \sin E_q x) \qquad ...(3.2)$$

Multiply both sides of (3.2) by

$$(\cos E_p x + \frac{h}{E_p} \sin E_p x)$$

then integrate with respect to x from 0 to L, and use (1.4), (2.1) and (2.2) to obtain

$$R_p = 2LE_p^2\,[(E_p^2 + h^2)L + 2h]^{-1}\,2^{1-\sigma-2\rho s-2\rho' s'}$$

$$\sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} F_{n,s}\,y^s \sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!} F_{n',s'}\,y'^{s'}$$

$$(\cos \frac{E_p \pi}{2} + \frac{h}{E_p} \sin \frac{E_p \pi}{2})\,H^{0,\lambda+1 : (u',v');...;(u^{(r)},v^{(r)})}_{A+1, C+2 : [B',D'];...;[B^{(r)},D^{(r)}]}$$

$$\left(\begin{matrix} [1-\sigma-2\rho s-2\rho' s' : 2k_1,...,2k_r],\,[(a) : \theta',...,\theta^{(r)}] : \\ [(c) : \psi',...,\psi^{(r)}],\,[\frac{1-\sigma-2\rho s-2\rho' s' \pm E_p}{2} : k_1,...,k_r] : \end{matrix} \right.$$

$$\left. \begin{matrix} [(b') : \phi'];...;[(b^{(r)}) : \phi^{(r)}]\,;\,z_1\,4^{-k_1} \\ \vdots \\ [(d') : \delta'];...;[(d^{(r)}) : \delta^{(r)}]\,;\,z_r\,4^{-k_r} \end{matrix} \right) \qquad ...(3.3)$$

Substitute the values of R_p from (3.3) in (3.2), the desired solution (3.1) is established.

4. SPECIAL CASES AND APPLICATIONS

(*i*) For $r = 1$, the results in (2.1), (2.2) and (3.1) reduce to the following results involving Fox's *H*-function [5]

$$\int_0^L (\sin \frac{\pi x}{L})^{\sigma - 1} \sin \frac{E_p \pi x}{L} S_n^m \left[y (\sin \frac{\pi x}{L})^{2\rho} \right]$$

$$. S_{n'}^{m'} \left[y' (\sin \frac{\pi x}{L})^{2\rho'} \right] H_{B,D}^{u,v} \left[z (\sin \frac{\pi x}{L})^{2k} \middle| \begin{matrix} [(b) : \phi] \\ [(d) : \delta] \end{matrix} \right] dx$$

$$= L.2^{1 - \sigma - 2\rho s - 2\rho' s'} \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} F_{n,s} y^s \sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!}$$

$$.F_{n',s'} y'^{s'} \sin \frac{E_p \pi}{2} H_{B+1, D+2}^{u, v+1} \left[z\, 4^{-k} \middle| \begin{matrix} [1 - \sigma - 2\rho s - 2\rho' s' : 2k], [(b) : \phi] \\ [(d) : \delta], [\frac{1 - \sigma - 2\rho s - 2\rho' s' \pm E_p}{2} : k] \end{matrix} \right] \quad ...(4.1)$$

where $\rho > 0$, $\rho' > 0$, $k > 0$, $\text{Re}\,(\sigma + kd_j/\delta_j > 0$, $1 \le j \le u$, m and m' are arbitrary positive integers and the coefficients $F_{n,s}$ $(n, s \ge 0)$ and $F_{n',s'}$ $(n', s' \ge 0)$ are arbitrary constants, real or complex.

$$(T = \sum_1^v \phi_i - \sum_{v+1}^B \phi_i + \sum_1^u \delta_i - \sum_{u+1}^D \delta_i)$$

$$\int_0^L (\sin \frac{\pi x}{L})^{\sigma - 1} \cos \frac{E_p \pi x}{L} S_n^m \left[y \sin \frac{\pi x}{L})^{2\rho} \right]$$

$$. S_{n'}^{m'} \left[y' (\sin \frac{\pi x}{L})^{2\rho'} \right] H_{B,D}^{u,v} \left[z (\sin \frac{\pi x}{L})^{2k} \middle| \begin{matrix} [(b) : \phi] \\ [(d) : \delta] \end{matrix} \right] dx$$

$$= L.2^{1 - \sigma - 2\rho s - 2\rho' s'} \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} F_{n,s} y^s F_{n,s} \sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!}$$

$$.F_{n',s'} y'^{s'} \cos \frac{E_p \pi}{2} H_{B+1, D+2}^{u, v+1} \left[z 4^{-k} \middle| \begin{matrix} [1 - \sigma - 2\rho s - 2\rho' s' : 2k], [(b) : \phi] \\ [(d) : \delta], [\frac{1 - \sigma - 2\rho s - 2\rho' s' \pm E_p}{2} : k] \end{matrix} \right] \quad ...(4.2)$$

where $\rho > 0$, $\rho' > 0$, $k > 0$, $\text{Re}\,(\sigma + k\, d_j/\delta_j > 0$, $1 < j \le u$,

$|\arg (z)| < T, \pi/2$, $T > 0$, m and m' are arbitrary positive integers and the coefficients $F_{n,s}$ $(n, s \ge 0)$ and $F_{n',s'}$ $(n', s' \ge 0)$ are arbitrary constants, real or complex.

$$\theta(x,t)=\frac{2L}{2^{\sigma+2\rho s+2\rho's'-1}}\sum_{q=1}^{\infty}\sum_{s=0}^{[n/m]}\frac{(-n)_{ms}}{s!}F_{n,s}\,y^{s}$$

$$\sum_{s'=0}^{[n'/m']}\frac{(-n')_{m's'}}{s'!}F_{n',s'}\,y'^{s'}\qquad\frac{E_q^2}{[(E_q^2+h^2)L+2h]}e^{-\mu E_q^2 t}$$

$$(\cos E_q x+\frac{h}{E_q}\sin E_q x)\,(\cos\frac{E_q\pi}{2}+\frac{h}{E_q}\sin\frac{E_q\pi}{2})$$

$$H^{u,\,v+1}_{B+1,\,D+2}\left[z4^{-k}\,\middle|\,\begin{matrix}[1-\sigma-2\rho s-2\rho's':2k],\ [(b):\phi]\\ [(d):\delta]\ [\frac{1-\sigma-2\rho s-2\rho's'\pm E_q}{2}:k]\end{matrix}\right]\quad\ldots(4.3)$$

where $\rho>0$, $\rho'>0$, $k>0$, $\mathrm{Re}\,(\sigma+kd_j/\delta_j)>0$, $1\le j\le u$, $|\arg(z)|<T\pi/2$, $T>0$, m and m' are arbitrary positive integer and the coefficients $F_{n,s}$ $(n,s\ge 0)$ and $F_{n',s'}$ $(n',s'\ge 0)$ are arbitrary constants real or complex.

(*ii*) Letting $n\to 0$, $n'\to 0$ and $y=y'=1$ the results in (2.1), (2.2), (3.1), (4.1), (4.2) and (4.3) follow as particular cases of results recently obtained by Chaurasia ([1], eqns. (16) and (17), p.79, eqn. (18), p.80, eqn. (21), p.81 and eqns. (22) and (23), p.82).

(*iii*) Taking $n'\to 0$, $y=y'=1$ the results in (2.1), (2.2), (3.1), (4.1), (4.2) and (4.3) follow as particular cases of the results recently obtained by Chaurasia and Girdhari Lal in [2].

(*iv*) By applying our results derived in (2.1), (2.2) and (2.3) to the case of Hermite polynomials ([10], p.106, eqn. (5.5.4) and [9], p.158) by setting

$$S_n^2[x]\to x^{n/2}H_n\left(\frac{1}{2\sqrt{x}}\right)$$

in which case $m=2$, $F_{n,s}=(-1)^s$ and also letting $m'=2$, $F_{n',s'}=(-1)^{s'}$, we have the following interesting consequences of the main result :

$$\int_0^L(\sin\frac{\pi x}{L})^{\sigma-1}\sin\frac{E_p\pi x}{L}\left[y\,(\sin\frac{\pi x}{L})^{2\rho}\right]^{n/2}$$

$$.\,H_n\left[\frac{1}{2\sqrt{y\,(\sin\frac{\pi x}{L})^{2\rho}}}\right]\left[y'\,(\sin\frac{\pi x}{L})^{2\rho'}\right]^{n'/2}$$

$$. H_{n'}\left[\frac{1}{2\sqrt{y'(\sin\frac{\pi x}{L})^{2\rho'}}}\right] H^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\begin{pmatrix} z_1(\sin\frac{\pi x}{L})^{2k_1} \\ \vdots \\ z_r(\sin\frac{\pi x}{L})^{2k_r}\end{pmatrix} dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho's'}\sum_{s=0}^{[n/2]}\frac{(-n)_{2s}}{s!}(-y)^s\sum_{s'=0}^{[n'/2]}\frac{(-n')_{2s'}}{s'!}$$

$$(-y')^{s'}\sin\frac{E_p\pi}{2} H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+2:[B',D'];\ldots;[B^{(r)},D^{(r)}]}$$

$$\left(\begin{matrix}[1-\sigma-2\rho s-2\rho's':2k_1,\ldots,2k_r],[(a):\theta',\ldots,\theta^{(r)}]: \\ [(c):\psi',\ldots,\psi^{(r)}],[\frac{1-\sigma-2\rho s-2\rho's'\pm E_p}{2}:k_1,\ldots,k_r]:\end{matrix}\right.$$

$$\left.\begin{matrix}(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(\rho)}];z_1 4^{-k_1} \\ \vdots \\ [(d'):\delta;\ldots;[(d^{(r)}):\delta^{(r)}];z_r 4^{-k_r}\end{matrix}\right) \quad \ldots(4.4)$$

which holds true under the same conditions as those required for (2.1)

$$\int_0^L(\sin\frac{\pi x}{L})^{\sigma-1}\cos\frac{E_p\pi x}{L}\left[y(\sin\frac{\pi x}{L})^{2\rho}\right]^{n/2}$$

$$.H_n\left[\frac{1}{2\sqrt{y(\sin\frac{\pi x}{L})^{2\rho}}}\right]\left[y'(\sin\frac{\pi x}{L})^{2\rho'}\right]^{n'/2} H_{n'}\left[\frac{1}{2\sqrt{y'(\sin\frac{\pi x}{L})^{2\rho'}}}\right]$$

$$H^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\begin{pmatrix} z_1(\sin\frac{\pi x}{L})^{2k_1} \\ \vdots \\ z_r(\sin\frac{\pi x}{L})^{2k_r}\end{pmatrix} dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho's'}\sum_{s=0}^{[n/2]}\frac{(-n)_{2s}}{s!}(-y)^s\sum_{s'=0}^{[n'/2]}\frac{(-n')_{2s'}}{s'!}$$

$$(-y')^{s'}\cos\frac{E_p\pi}{2} H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+2:[B',D'];\ldots;[B^{(r)},D^{(r)}]}$$

$$\left(\begin{matrix} [1-\sigma-2\rho s-2\rho' s' : 2k_1 ,..., 2k_r], [(a):\theta' ,..., \theta^{(r)}] : \\ [(c):\psi' ,..., \psi^{(r)}], [\dfrac{1-\sigma-2\rho s-2\rho' s' \pm E_p}{2} : k_1 ,..., k_r] : \end{matrix} \right.$$

$$\left. \begin{matrix} (b'):\phi'] ;...; [(b^{(r)}):\phi^{(\rho)}] ; z_1\, 4^{-k_1} \\ \vdots \\ [(d'):\delta'] ;...; [(d^{(r)}):\delta^{(r)}]; z_r\, 4^{-k_r} \end{matrix} \right) \quad ...(4.5)$$

valid under some conditions as needed for (2.2)

$$\theta(x,t) = \frac{2L}{2^{\sigma+2\rho s+2\rho' s'-1}} \sum_{q=1}^{\infty} \sum_{s=0}^{[n/2]} \sum_{s'=0}^{[n'/2']} \frac{(-n)_{2s}}{s!} (-y)^s$$

$$\frac{(-n')_{2s'}}{s'!} (-y')^{s'} \frac{E_q^2}{[(E_q^2+h^2)L+2h]} e^{-\mu E_q^2 t} (\cos E_q x + \frac{h}{E_q} \sin E_q x)$$

$$(\cos \frac{E_q \pi}{2} + \frac{h}{E_q} \sin \frac{E_q \pi}{2}) H^{0,\lambda+1:(u',v');...;(u^{(r)},v^{(r)})}_{A+1,C+2:[B',D'];...;[B^{(r)},D^{(r)}]}$$

$$\left(\begin{matrix} [1-\sigma-2\rho s-2\rho' s' : 2k_1 ,..., 2k_r], [(a):\theta' ,..., \theta^{(r)}] : \\ [(c):\psi' ,..., \psi^{(r)}], [\dfrac{1-\sigma-2\rho s-2\rho' s' \pm E_q}{2} : k_1 ,..., k_r] : \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'):\phi'] ;...; [(b^{(r)}):\phi^{(r)}] ; z_1 4^{-k_1} \\ \vdots \\ [(d'):\delta'] ;...; [(d^{(r)}):\delta^{(r)}] ; z_r\, 4^{-k_r} \end{matrix} \right) \quad ...(4.6)$$

which holds true under the same conditions as those required for (3.1)

(v) Taking $\lambda = A_i$, $u^{(i)} = 1$, $v^{(i)} = B^{(i)}$, $D^{(i)} = D^{(i)} + 1$, $\forall i = 1, 2, ..., r$ in (2.1), (2.2) and (3.1), we get

$$\int_0^L (\sin \frac{\pi x}{L})^{\sigma-1} \sin \frac{E_p \pi x}{L}\; S_n^m [(y \frac{\pi x}{L})^{2\rho}]$$

$$S_{n'}^{m'} \left[y' (\sin \frac{\pi x}{L})^{2\rho'} \right] F^{A:B';...;B^{(r)}}_{C:D';...;D^{(r)}} \left(\begin{matrix} [1-(a):\theta' ,..., \theta^{(r)}] : \\ [1-(c):\psi' ,..., \psi^{(r)}] : \end{matrix} \right.$$

$$\left. \begin{matrix} [1-(b'):\phi'] ;...; [1-(b^{(r)}):\phi^{(r)}]; \\ [1-(d'):\delta'] ;...; [1-(d^{(r)}):\delta^{(r)}]; \end{matrix} \begin{matrix} -z_1 (\sin \frac{\pi x}{L})^{2k_1} \\ \vdots \\ -z_r (\sin \frac{\pi x}{l})^{2k_r} \end{matrix} \right) dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho's'} \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} F_{n,s}\, y^s \sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!}$$

$$.F_{n',s'}\, y'^{s'} \sin\frac{E_p\pi}{2} \frac{\Gamma(\sigma+2\rho s+2\rho' s')}{\Gamma\left(\frac{\sigma+2\rho s+2\rho' s' \pm E_p+1}{2}\right)}$$

$$.F^{A+1:B';\ldots;B^{(r)}}_{C+2:D';\ldots;D^{(r)}}\left(\begin{array}{l}[\sigma+2\rho s+2\rho' s' : 2k_1,\ldots,2k_r],\\ [1-(c):\psi',\ldots,\psi^{(r)}],\end{array}\right.$$

$$[1-(a):\theta',\ldots,\theta^{(r)}]:[1-(b'):\phi'];\ldots;[1-(b^{(r)}):\phi^{(r)}];$$

$$[\frac{\sigma+2\rho s+2\rho' s' \pm E_p+1}{2} : k_1,\ldots,k_r]:[1-(d'):\delta'];\ldots;[1-(d^{(r)}):\delta^{(r)}]$$

$$\left. -z_1 4^{-k_1},\ldots,-z_r 4^{-k_r}\right) \qquad \ldots(4.7)$$

valid under the same conditions as obtainable from (2.1).

$$\int_0^L (\sin\frac{nx}{L})^{\sigma-1} \cos\frac{E_p\pi x}{L} S_n^m\left[y(\sin\frac{\pi x}{L})^{2\rho}\right]$$

$$S_{n'}^{m'}\left[y'(\sin\frac{\pi x}{L})^{2\rho'}\right] F^{A:b';\ldots;B^{(r)}}_{C:d';\ldots;D^{(r)}}\left(\begin{array}{l}[1-(a):\theta',\ldots,\theta^{(r)}:\\ [1-(c):\psi',\ldots,\psi^{(r)}]:\end{array}\right.$$

$$\left.\begin{array}{ll}[1-(b'):\phi'];\ldots;[1-(b^{(r)}):\phi^{(r)}]; & -z_1(\sin\frac{\pi x}{L})^{2k_1}\\ & \vdots \\ [1-(d'):\delta'];\ldots;[1-(d^{(r)}):\delta^{(r)}]; & -z_r(\sin\frac{\pi x}{L})^{2k_r}\end{array}\right) dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho's'} \sum_{s=0}^{[n/m]} \frac{(-n)_{ms}}{s!} F_{n,s}\, y^s \sum_{s'=0}^{[n'/m']} \frac{(-n')_{m's'}}{s'!}$$

$$.F_{n',s'}\, y'^{s'} \cos\frac{E_p\pi}{2} \frac{\Gamma(\sigma+2\rho s+2\rho' s')}{\Gamma\left(\frac{\sigma+2\rho s+2\rho' s' \pm E_p+1}{2}\right)}$$

$$.F^{A+1:B';\ldots;B^{(r)}}_{C+2:D';\ldots;D^{(r)}}\left(\begin{array}{l}[\sigma+2\rho s+2\rho' s' : 2k_1,\ldots,2k_r],[1-(a):\theta',\ldots,\theta^{(r)}]:\\ [1-(c):\psi',\ldots,\psi^{(r)}],[\frac{\sigma+2\rho s+2\rho' s' \pm E_p+1}{2}:k_1,\ldots,k_r]:\end{array}\right.$$

$$\left.\begin{array}{ll}[1-(b'):\phi'];\ldots;[1-(b^{(r)}):\phi^{(r)}]; & -z_1 4^{-k_1}\\ & \vdots \\ [1-(d'):\delta'];\ldots;[1-(d^{(r)}):\delta^{(r)}]; & -z_r 4^{-k_r}\end{array}\right) \qquad \ldots(4.8)$$

valid under the same conditions as need for (2.2)

$$\theta(x,t)=\frac{2L}{2^{\sigma+2\rho s+2\rho' s'-1}}\sum_{q=1}^{\infty}\sum_{s=0}^{[n/m]}\sum_{s'=0}^{[n'/m']}\frac{(-n)_{ms}}{s!}$$

$$.F_{n,s}\,y^{s}\frac{(-n')_{m',s'}}{s'!}F_{n',s'}\,y'^{s'}\frac{E_q^2}{[(E_q^2+h^2)L+2h]}e^{-\mu E_{q'}^2 t}$$

$$(\cos E_q x+\frac{h}{E_q}\sin E_q x)(\cos\frac{E_q\pi}{2}+\frac{h}{E_q}\sin\frac{E_q\pi}{2})$$

$$\frac{\Gamma(\sigma+2\rho s+2\rho' s')}{\Gamma\left(\frac{\sigma+2\rho s+2\rho' s'\pm E_q+1}{2}\right)}\frac{\prod_{j=1}^{A}\Gamma(1-a_j)\prod_{j=1}^{B'}\Gamma(1-b_j')\ldots\prod_{j=1}^{B^{(r)}}\Gamma(1-b_j^{(r)})}{\prod_{j=1}^{C}\Gamma(1-c_j)\prod_{j=1}^{D'}\Gamma(1-d_j')\ldots\prod_{j=1}^{D^{(r)}}\Gamma(1-d_j^{(r)})}$$

$$.F^{A+1:B';\ldots;B^{(r)}}_{C+2:D';\ldots;D^{(r)}}\left(\begin{matrix}[\sigma+2\rho s+2\rho' s':2k_1,\ldots,2k_r],[1-(a):\theta',\ldots,\theta^{(r)}]:\\ [1-(c):\psi',\ldots,\psi^{(r)}],\left[\frac{\sigma+2\rho s+2\rho' s'\pm E_q+1}{2}:k_1,\ldots,k_r\right]\end{matrix}\right.$$

$$\left.\begin{matrix}[1-(b'):\phi'];\ldots;[1-b^{(r)}:\phi^{(r)}];\\ [1-(d'):\delta'];\ldots;[1-d^{(r)}:\delta^{(r)}];\end{matrix}\begin{matrix}-z_1 4^{-k_1}\\ \vdots\\ -z_r 4^{-k_r}\end{matrix}\right)\qquad\ldots(4.9)$$

valid under the same conditions as obtainable from (3.1).

(*vi*) For the Laguerre polynomials ([10]; p.101, eqn. 1.5, 1.6) and ([9], p.158) setting $S_n^1(x)\rightarrow L_n^{(\alpha')}(x)$,

in which case $m=1$ and $F_{n,s}=\binom{n+\alpha'}{n}\frac{1}{(\alpha'+1)_s}$

and also taking $m'=1$ and $F_{n',s'}=\binom{n'+\alpha''}{n'}\frac{1}{(\alpha''+1)_{s'}}$

the results in (2.1), (2.2) and (3.1) reduce to the following formulae :

$$\int_0^L(\sin\frac{\pi x}{L})^{\sigma-1}L_n^{\alpha'}\left[y(\sin\frac{\pi x}{L})^{2\rho}\right]L_{n'}^{\alpha''}\left[y'(\sin\frac{\pi x}{L})^{2\rho'}\right]$$

$$.\sin\frac{E_p\pi x}{L}\,H^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}z_1(\sin\frac{\pi x}{L})^{2k_1}\\ \vdots\\ z_r(\sin\frac{\pi x}{L})^{2k_r}\end{matrix}\right)dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho' s'} \sum_{s=0}^{n} \binom{n+\alpha'}{n-s} \frac{(-y)^s}{s!} \sum_{s'=0}^{n'} \binom{n'+\alpha''}{n'-s'}$$

$$\cdot \frac{(-y')^{s'}}{s'!} \sin \frac{E_p \pi}{2} H^{0,\lambda+1:(u',v');...;(u^{(r)},v^{(r)})}_{A+1,C+2:[B',D'];...;[B^{(r)},D^{(r)}]}$$

$$\left(\begin{matrix} [1-\sigma-2\rho s-2\rho' s' : 2k_1, ..., 2k_r], [(a):\theta', ..., \theta^{(r)}] : \\ [(c):\psi', ..., \psi^{(\rho)}], \left[\frac{1-\sigma-2\rho s-2\rho' s' \pm E_p}{2} : k_1, ..., k_r \right] : \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'):\phi'];...;[(b^{(r)}):\phi^{(r)}]; z_1 4^{-k_1} \\ \vdots \\ [(d'):\delta'];...;[(d^{(r)}):\delta^{(r)}]; z_r 4^{-k_r} \end{matrix} \right) \quad ...(4.10)$$

valid under the same conditions as needed for (2.1)

$$\int_0^L (\sin \frac{\pi x}{L})^{\sigma-1} \cos \frac{E_p \pi x}{L} L_n^{\alpha'} \left[y (\sin \frac{\pi x}{L})^{2\rho} \right] L_{n'}^{\alpha''} \left[y' (\sin \frac{\pi x}{L})^{2\rho'} \right]$$

$$H^{0,\lambda:(u',v');...;(u^{(r)},v^{(r)})}_{A,C:[B',D'];...;[B^{(r)},D^{(r)}]} \left(\begin{matrix} z_1 (\sin \frac{\pi x}{L})^{2k_1} \\ \vdots \\ z_r (\sin \frac{\pi x}{L})^{2k_r} \end{matrix} \right) dx$$

$$= L.2^{1-\sigma-2\rho s-2\rho' s'} \sum_{s=0}^{n} \binom{n+\alpha'}{n-s} \frac{(-y)^s}{s!} \sum_{s'=0}^{n'} \binom{n'+\alpha''}{n'-s'} \frac{(-y')^{s'}}{s'!} \cos \frac{E_p \pi}{2}$$

$$H^{0,\lambda+1:(u',v');...;(u^{(r)},v^{(r)})}_{A+1,C+2:[B',D'];...;(B^{(r)},D^{(r)})]}$$

$$\left(\begin{matrix} [1-\sigma-2\rho s-2\rho' s' : 2k_1, ..., 2k_r], [(a):\theta', ..., \theta^{(r)}] : \\ [(c):\psi', ..., \psi^{(r)}], [\frac{1-\sigma-2\rho s-2\rho' s' \pm E_p}{2} : k_1, ..., k_r] : \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'):\phi'];...;[(b^{(r)}):\phi^{(r)}]; z_1 4^{-k_1} \\ \vdots \\ [(d'):\delta'];...;[(d^{(r)}):\delta^{(r)}]; z_r 4^{-k_r} \end{matrix} \right) \quad ...(4.11)$$

valid under the same conditions as obtainable from (2.2)

$$\theta(x,t) = \frac{2L}{2^{\sigma+2\rho s+2\rho' s'-1}} \sum_{q=1}^{\infty} \sum_{s=0}^{n} \binom{n+\alpha'}{n-s} \frac{(-y)^s}{s!}$$

$$\sum_{s'=0}^{n'} \binom{n'+\alpha''}{n'-s'} \frac{(-y')^{s'}}{s'!} \frac{E_q^2}{[(E_q^2+h^2)L+2h]} e^{-\mu E_q^2 t}$$

$$(\cos E_q x + \frac{h}{E_q} \sin E_q x)(\cos \frac{E_q \pi}{2} + \frac{h}{E_q} \sin \frac{E_q \pi}{2})$$

$$H^{0,\,\lambda+1\,:\,(u',\,v')\,;\ldots;\,(u^{(r)},\,v^{(r)})}_{A+1,\,C+2\,:\,[B',\,D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}$$

$$\left(\begin{matrix} [1-\sigma-2\rho s-2\rho' s' : 2k_1, \ldots, 2k_r], [(a) : \theta', \ldots, \theta^{(r)}] : \\ [(c) : \psi', \ldots, \psi^{(r)}], [\frac{1-\sigma-2\rho s-2\rho' s' \pm E_q}{2} : k_1, \ldots, k_r] : \end{matrix}\right.$$

$$\left.\begin{matrix} [(b') : \phi'] ;\ldots; [b^{(r)} : \phi^{(r)}] ; z_1 4^{-k_1} \\ \vdots \\ [(d') : \delta'] ;\ldots; [d^{(r)} : \delta^{(r)}] ; z_r 4^{-k_r} \end{matrix}\right) \qquad \ldots(4.12)$$

valid under the same conditions as obtainable from (3.1).

The importance of our resutls lies in its manifold generality. In view of the generality of the multivariable H-function, on specializing the various parameters and variables, from our results, several integrals and solutions involving a remarkably wide variety of useful functions (or products of several such functions), which are expressible in terms of E, F, G and H functions of one and several variables.

Secondly, on suitably specializing the coefficients $F_{n,s}$ and also the coefficients $F_{n',s'}$ and making a free use of the special cases of $S_n^m[x]$ listed by Srivastava and Singh [9], our results can be reduced to a large number of integerals and the solution of the problem involving the products of generalized Hermite polynomials, Jacobi polynomials, Brafman polynomials, Gould-Hopper polynomials and their various combinations. Thus, the result derived in this paper would at once yield a very large number of results, involving a large variety of polynomials and various special functions occuring in the problems of mathematical analysis, applied mathematics and mathematical physics.

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