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A GENERAL FRACTIONAL INTEGRAL OPERATOR, A GENERAL CLASS OF MULTIVARIABLE POLYNOMIALS AND THE MULTIVARIABLE *H*-FUNCTION

By

S.P. Goyal and R.S. Pareek
Department of Mathematics, University of Rajasthan,
Jaipur- 302004, Rajasthan, India

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ABSTRACT

The main object of this paper is to derive a fractional integral operator (involving a generalized polynomial set) of the product of a general class of multivariable polynomials and the *H*-function of several complex variables. On account of general nature of the operator, the multivariable polynomials and the *H*-function of several complex variables, our formula can be shown to yield a number of interesting (known or new) results. Some new special cases of the formula are mentioned briefly.

1. INTRODUCTION

(a) General Fractional Integral Operator

The familier fractional integral operator (FIO) is defined and represented as follows :

$$ {}_cR_x^{\mu}\,\{f(x)\} = \frac{1}{\Gamma(\mu)} \int_c^x (x-t)^{\mu-1} f(t)\,dt,\ Re(\mu) > 0 \qquad ...(1.1)$$

The *FIO* (1.1) defines the classical Riemann-Liouville *FIO* of order μ when $c = 0$ and when $c \to \infty$ it may be identified with the definition of the familier Weyl *FIO* of order μ. A number of generalizations of *FIO* are proposed recently. Agrawal, Pareek and Saigo [1,p. 55, Eq. (1.2)] have recently introduced and studied a generalization of *FIO* (1.1) which is defined by representing in the following manner :

$$ {}_cR_{x;\,r,\,s,\,q;\,m,\,k,\,l}^{\rho;\,\alpha,\,\beta,\,\tau;\,A,\,B,\,\mu}\,\{f(x)\}$$

$$= \frac{1}{\Gamma(\mu)} \int_c^x (x-t)^{\mu-1} S_n^{\alpha,\beta,\tau}[z(1-\tfrac{t}{x})^\rho; r, s, q, A, B, m, k, l]\, f(t)\, dt \qquad ...(1.2)$$

where Re(μ) > 0 and $S_n^{\alpha,\beta,\tau}[z]$ stands for the generalized polynomial set due to Raizada [6, p.64, Eq. (2.1.8)].

The explicit form of this generalized polynomial set ([6,p.71, Eq. (2.3.4)] : see also [1, p.56, Eq. (1.5)]) is

$$S_n^{\alpha,\beta,\tau}[x; r, s, q, A, B, m, k, l] = B^{qn} x^{l(m+n)} (l - \tau x^r)^{sn}$$

$$l^{m+n} \sum_{v=0}^{m+n} \sum_{u=0}^{v} \sum_{j=0}^{m+n} \sum_{i=0}^{j} \frac{(-1)^j (-j)_i (\alpha)_j (-v)_{u} (-\alpha - qn)_i}{i!\, j!\, u!\, v!\, (1-\alpha-j)_i}$$

$$\left(-\frac{\beta}{\tau} - sn\right)_v \left(\frac{i+k+ru}{l}\right)_{m+n} \left(\frac{-\tau x^r}{1-\tau x^r}\right)^v \left(\frac{Ax}{B}\right)^j \qquad ...(1.3)$$

In this paper we shall study the following special case of *FIO* (1.2) defined and represented in the following manner :

$${}_cR_{x:\,r,\,q;\,m,\,k,\,l}^{\rho;\,\alpha,\,\beta,\,0;\,1,\,0,\,\mu}\{f(x)\} = \frac{1}{\Gamma(\mu)} \int_x^c (x-t)^{\mu-1}$$

$$S_n^{\alpha,\beta,0}[z(1-t/x)^\rho; r, q, 1, 0, m, k, l]\, f(t)\, dt \qquad ...(1.4)$$

where $Re(\mu) > 0$ and

$$S_n^{\alpha,\beta,0}[x; r, q, 1, 0, m, k, l] = x^{qn+l(m+n)}\, l^{m+n}$$

$$\sum_{v=0}^{m+n} \sum_{u=0}^{v} \frac{(-v)_u}{v!\, u!} \left(\frac{\alpha+qn+k+ru}{l}\right)_{m+n} (\beta x^r)^v \qquad ...(1.5)$$

It may be pointed out here that *FIO* (1.4) is sufficiently general in nature as it involves a generalized polnomial set which unifies and extends a large number of classical polynomials introduced and studied by various research workers such as Chatterjea [2], Gould and Hopper [3], Krall and Frink [4] and Singh and Shrivastava [8], etc.

(b) A General class of multivariable Polynomials

Srivastava and Garg [10,p.685] introduced and studied a general class of multivariable polynomial in the following manner

$$S_n^{m_1,\ldots,m_s}[x_1, \ldots, x_s]$$

$$= \sum_{k_1, \ldots, k_s = 0}^{m_1 k_1 + \ldots + m_s k_s \leq n} (-n)_{m_1 k_1 + \ldots + m_s k_s} A(n\,; k_1, \ldots, k_s) \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_s^{k_s}}{k_s!} \quad \ldots(1.6)$$

where $m_1, \ldots, m_s$ are arbitrary positive integers and the coefficients $A(n; k_1, \ldots, k_s)$ $(n, k_i \geq 0, i = 1, \ldots, s)$ are arbitrary constant real or complex.

(c) The Multivariable *H*-function

The multivariable H-function occurring in this paper has been defined by Srivastava and Panda [13]. We shall use the following contracted notation (Srivastava et al. [12, p.251, Eq.(C.1)] :

$$H[z_1, \ldots, z_p]$$

$$= H^{O,\, N\, ;\, M_1,\, N_1\, ;\ldots;\, M_p,\, N_p}_{P,\, Q\, ;\, P_1,\, Q_1 ;\ldots;\, P_p,\, Q_p} \left[\begin{matrix} z_1 \\ \vdots \\ z_p \end{matrix} \middle| \begin{matrix} (a_j; A_j', \ldots, A_j^{(p)})_{1,\, P}; \\ (b_j; B_j', \ldots, B_j^{(p)})_{1,\, Q}; \end{matrix} \right.$$

$$\left. \begin{matrix} (c_j', C_j')_{1,\, P_1}; \ldots; (c_j^{(p)}, C_j^{(p)})_{1,\, P_p} \\ (d_j', D_j')_{1,\, Q_1}; \ldots; (d_j^{(p)}, D_j^{(p)})_{1,\, Q_p} \end{matrix} \right] \quad \ldots(1.7)$$

to denote the H-function of p-complex variables here all the capital letters are assumed to be positive real numbers for the standardization purposes; the definition of multivariable H-function given by (1.7) will, however, be meaningful even if some of these quantities are zero. The details of this function can be found in paper and book referred to above.

2. MAIN RESULT

$${}_c R^{\rho;\, \alpha,\, \beta,\, 0;\, 1,\, 0,\, \mu}_{x;\, r,\, q;\, m,\, k,\, l} \{[(x-a)^v + \xi]^{\sigma} (x-a)^{\lambda}$$

$$S_{n'}^{m_1, \ldots, m_s} [y_1 (x-a)^{\lambda_1} \{(x-a)^v + \xi\}^{-\tau_1}, \ldots, y_s (x-a)^{\lambda_s} \{(x-a)^v + \xi\}^{-\tau_s}]$$

$$H[z_1 (x-a)^{\rho_1} \{(x-a)^v + \xi\}^{-\sigma_1}, \ldots, z_p (x-a)^{\rho_p} \{(x-a)^v + \xi\}^{-\sigma_p}]\}$$

$$= \frac{1}{\Gamma(\mu)} \sum_{v=0}^{m+n} \sum_{u=0}^{v} \sum_{k_1, \ldots, k_s = 0}^{m_1 k_1 + \ldots + m_s k_s \leq n'} \sum_{w=0}^{\infty} \sum_{\delta=0}^{\infty} \frac{(-1)^{w+\delta}}{w!\, \delta!}$$

$$\frac{B\,(m, n, u, v)\, D(n', k_i, y_i)\xi}{(\mu \cdot \; [illegible] \; \cdot \delta)}$$

$$(x-a)^{\lambda + [illegible] \; \delta + \sum_{i=1}^{s} \lambda_i k_i} (x-c)^{\mu + A + \delta}\, x^{-A}$$

$$H^{0,N+2:M_1,N_1;\dots;M_p,N_p}_{P+2,Q+2:P_1,Q_1;\dots;P_p,Q_p}\left[\begin{array}{c|} z_1(x-a)^{\rho_1}\xi^{-\sigma_1} \\ \vdots \\ z_p\,(x-a)^{\rho_p}\,\xi^{-\sigma_p}\end{array}\right.$$

$$(-\lambda-vw-\sum_{i=1}^{s}\lambda_i k_i;\ \rho_1,\dots,\rho_p),$$
$$(b_j;B_j',\dots,B_j^{(p)})_{1,Q'}$$

$$(1-w+\sigma-\sum_{i=1}^{s}\tau_i k_i;\ \sigma_1,\dots,\sigma_p),\ (a_j;A_j',\dots,A_j^{(p)})_{1,p}:$$

$$(\delta-\lambda-vw-\sum_{i=1}^{s}\lambda_i k_i;\ \rho_1,\dots,\rho_p),\ (1+\sigma-\sum_{i=1}^{s}\tau_i k_i;\ \sigma_1,\dots,\sigma_p):$$

$$\left.\begin{array}{l}(c_j',C_j')_{1,P_1};\dots;(c_j^{(p)},C_j^{(p)})_{1,P_p}\\ (d_j',D_j')_{1,Q_1};\dots;(d_j^{(p)},D_j^{(p)})_{1,Q_p}\end{array}\right] \qquad \dots(2.1)$$

where for convienience

$$B(m,n,u,v)=\frac{(-v)_u}{v!\,u!}\left(\frac{\alpha+qn+k+ru}{l}\right)_{m+n} l^{m+n}\,\beta^{v}\,z^{A/\rho} \qquad \dots(2.2)$$

$$D(n',k_i,y_i)=(-n')_{m_1k_1+\dots+m_sk_s}\,A(n';k_1,\dots,k_s)\,\frac{y_1^{k_1}}{k_1!}\cdots\frac{y_s^{k_s}}{k_s!} \qquad \dots(2.3)$$

$$A=qn\rho+l\rho(m+n)+rv\rho \qquad \dots(2.4)$$

and (in addition to the appropriate convergence and existence conditions [12,p.252]), we assume that

$$\min\{v,\rho,\lambda,\sigma,\lambda_i,\tau_i,\rho_j,\sigma_j\}>0\ (i=1,\dots,s,j=1,\dots,p),$$

$$|\arg(x-a)^{v}/\xi|<\pi,\ c\ge a\ge 0,$$

$$\mathrm{Re}\left[\lambda+\sum_{i=1}^{s}\lambda_i k_i+\sum_{j=1}^{p}\rho_j\zeta_i\right]>-1\ (\zeta_j=\min_{1\le k\le m_j}[Re(d_k^{(j)}/D_k^{(j)})],$$

$\mathrm{Re}\,[\mu+qn\rho+l\rho(m+n)+rv\rho]>0 \quad (v=1,\dots,m+n)$ and the multiple series occuring on right hand side of (2.1) is absolutely convergent.

Derivation of formula (2.1):

Using the defintion of *FIO* (1.4) in (2.1), we find that

$$\text{L.H.S. of (2.1)}=\frac{1}{\Gamma(\mu)}\int_c^x (x-t)^{\mu-1}\,\{(t-a)^{\lambda}\,[(t-a)^{v}+\xi]^{\sigma}$$

$$S_n^{\alpha,\beta,0}[z(1-t/x); r, q, 1, 0, m, k, l]$$

$$S_{n'}^{m_1,\ldots,m_s}[y_1(t-a)^{\lambda_1}\{(t-a)^v+\xi\}^{-\tau_1},\ldots, y_s(t-a)^{\lambda s}\{(t-a)^v+\xi\}^{-\tau_s}]$$

$$H[z_1(t-a)^{\rho_1}\{(t-a)^v+\xi\}^{-\sigma_1},\ldots, z_p(t-a)^{\rho_p}(t-a)^v+\xi\}^{-\sigma_p}]\}dt \quad \ldots(2.5)$$

Now replacing in (2.5) the generalized polynomial set $S_n^{\alpha,\beta,0}[z(1-t/x)^\rho : r, q, 1, 0, m, k, l]$ the general class of multivariable polynomials,

$$S_{n'}^{m_1,\ldots,m_s}[y_1(t-a)^{\lambda_1}\{(t-a)^v+\xi\}^{-\tau_1},\ldots y_s(t-a)^{\lambda_s}\{(t-a)^v+\xi\}^{-\tau_s}]$$

by their series given by (1.5) and (1.6) respectively and multivariable H-function by its Mellin-Barnes contour integral given by (1.7) and interchanging the order of integeration and summation, we get

L.H.S. of (2.1)

$$=\frac{1}{\Gamma(\mu)}\sum_{v=0}^{m+n}\sum_{u=0}^{v}\sum_{k_1,\ldots,k_s=0}^{m_1k_1+\ldots+m_sk_s\le n'} x^{-A}$$

$$B(m,n,u,v)\,D(n',k_i,y_i)\frac{1}{(2\pi i)^p}\int_{L_1}\ldots,\int_{L_p}\phi(\eta_1,\ldots,\eta_p)$$

$$\prod_{j=1}^{p}\{\theta_j(\eta_j)(z_j)^{\eta_j}\}\,\{\int_c^x (x-t)^{\mu+A-1}(t-a)^{\lambda+\sum_{i=1}^{s}\lambda_ik_i+\sum_{j=1}^{p}\rho_j\eta_j}$$

$$[(t-a)^v+\xi]^{\sigma-\sum_{i=1}^{s}\tau_ik_i-\sum_{j=1}^{p}\sigma_j\eta_j}dt\}\,d\eta_1,\ldots,d\eta_p \quad \ldots(2.6)$$

Applying the binomial expansion

$$(x+\xi)^\lambda=\xi^\lambda\sum_{w=0}^{\infty}\frac{(-1)^w(-\lambda)_w}{w!}\left(\frac{x}{\xi}\right)^w,\left(\left|\frac{x}{\xi}\right|<1\right) \quad \ldots(2.7)$$

Changing the order of integration and summation and using the well-known formula (cf,[7,p.22]) :

$$\int_c^x (x-t)^{\mu-1}(t-a)^\lambda dt$$

$$=(x-c)^\mu(x-a)^\lambda\sum_{\delta=0}^{\infty}\frac{(-1)^\delta\Gamma(\lambda+1)}{(\mu+\lambda)\,\delta!\,\Gamma(\lambda-\delta+1)}\left(\frac{x-c}{x-a}\right)^\delta \quad \ldots(2.8)$$

where Re $(\mu)>0$, Re$(\lambda)>-1$ and $c\ge a\ge 0$, we obtain

$$\text{L.H.S. of (2.1)} = \frac{1}{\Gamma(\mu)} \sum_{v=0}^{m+n} \sum_{u=0}^{v} \sum_{k_1,\ldots,k_s=0}^{m_1k_1+\ldots+m_sk_s \le n'} \sum_{w=0}^{\infty} \sum_{\delta=0}^{\infty}$$

$$B(m, n, u, v)\, D(n', k_i, y_i)\, \frac{(-1)^{w+\delta}}{w!\,\delta!}\, \xi^{\sigma - w - \sum_{i=1}^{s} \tau_i k_i}$$

$$x^{-A} (\mu + A + \delta)^{-1} (x-c)^{\mu+\delta+A} (x-a)^{\lambda + vw - \delta + \sum_{i=1}^{s} \lambda_i k_i}$$

$$\left(\frac{1}{2\pi i}\right)^{p} \int_{L_1} \cdots \int_{L_p} \frac{\Gamma\left(w - \sigma + \sum_{i=1}^{s} \tau_i k_i + \sum_{j=1}^{p} \sigma_j \eta_j\right)}{\Gamma\left(-\sigma + \sum_{i=1}^{s} \tau_i k_i + \sum_{j=1}^{p} \sigma_j \eta_j\right)}$$

$$\frac{\Gamma\left(1 + \lambda + vw + \sum_{i=1}^{s} \lambda_i k_i + \sum_{j=1}^{p} \rho_j \eta_j\right)}{\Gamma\left(1 - \delta + \lambda + vw + \sum_{i=1}^{s} \lambda_i k_i + \sum_{j=1}^{p} \rho_j \eta_j\right)}\, \phi(\eta_1, \ldots, \eta_p)$$

$$\left[\prod_{j=1}^{p} \theta_j(\eta_j)\, z_j^{\eta_j}\right] \xi^{-\sum_{j=1}^{p} \sigma_j \eta_j} (x-a)^{\sum_{j=1}^{p} \rho_j \eta_j}\, ; d\eta_1, \ldots, d\eta_p \qquad \ldots(2.9)$$

Interpreting the multiple Mellin-Barnes contour integral as an $\bar{H}$-function of p-variables, we led finally to the fractional integral formula (2.1) under the (sufficient) conditions of validity stated already with (2.1).

3. SPECIAL CASES

The importance of our generalized fractional integral formula lies in its manifold generality. First of all our *FIO* (1.4) involves the generalized polynomial set which can be reduced to a large number of polynomials (see e.g.[6]). Secondly, the general class of multi-variable polynomials involved in (2.1) can be reduced to a large spectrum of polynomials of one and several variables. Thirdly, the formula (2.1) can be suitably applied to derive the corresponding results involving a remarkably wide variety of potentially useful functions (or product of several such functions) which are expressible in terms of the *E, G* and *H* functions of one, two or more variables. Our main result thus provides unification and extension of various (known or new) results on *FIO*. For the sake of illustration, we mention the following interesting special cases of (2.1):

(i) Taking $a = c$ in (2.1) and using the well-known Gauss summation theorem to sum the δ-series, we get after a little simplification

$$
{}_cR^{\rho;\,\alpha,\,\beta,\,0;\,1,\,0,\,\mu}_{x;\,r,\,q;\,m,\,k,\,l}\ \{(x-c)^{\lambda}\,[(x-c)^{v}+\xi]^{\sigma}
$$

$$
S_{n'}^{m_1,\ldots,m_s}\,[y_1(x-c)^{\lambda_1}\{(x-c)^{v}+\xi\}^{-\tau_1},\ldots,y_s(x-c)^{\lambda_s}\{(x-c)^{v}+\xi\}^{-\tau_s}]
$$

$$
H[z_1(x-c)^{\rho_1}\{(x-c)^{v}+\xi\}^{-\sigma_1},\ldots,z_p(x-c)^{\rho_p}\{(x-c)^{v}+\xi\}^{-\sigma_p}]\}
$$

$$
=\frac{1}{\Gamma(\mu)}\sum_{v=0}^{m+n}\sum_{u=0}^{v}\sum_{k_1,\ldots,k_s=0}^{m_1k_1+\ldots+m_sk_s\le n'}\sum_{w=0}^{\infty}\frac{(-1)^w}{w!}B(m,n,u,v)
$$

$$
D(n',k_i,y_i)\,\Sigma\,(\mu+A)x^{-A}\,\xi^{\sigma-w-\sum_{i=1}^{s}\tau_ik_i}\,(x-c)^{\mu+\lambda+A+vw+\sum_{i=1}^{s}\lambda_ik_i}
$$

$$
H^{0,\,N+2\,:\,M_1,\,N_1\,;\ldots;\,M_p,\,N_p}_{P+2,\,Q+2\,:\,P_1,\,Q_1\,;\ldots;\,P_p,\,Q_p}\left[\begin{array}{c} z_1\,(x-c)^{\rho_1}\,\xi^{-\sigma_1} \\ \vdots \\ z_p\,(x-c)^{\rho_p}\,\xi^{-\sigma_p}\end{array}\right|
$$

$$
(-\lambda-vw-\sum_{i=1}^{s}\lambda_ik_i\,;\,\rho_1,\ldots,\rho_p),
$$

$$
(b_j\,;\,B_j',\ldots,B_j^{(p)})_{1,Q}
$$

$$
(1-w+\sigma-\sum_{i=1}^{s}\tau_ik_i\,;\,\sigma_1,\ldots,\sigma_p),\,(a_j\,;\,A_j',\ldots,A_j^{(p)})_{1,P}\,:
$$

$$
(-\mu-A-\lambda-vw-\sum_{i=1}^{s}\lambda_ik_i\,;\,\rho_1,\ldots,\rho_p),\,(1+\sigma-\sum_{i=1}^{s}\tau_ik_i\,;\,\sigma_1,\ldots,\sigma_p)
$$

$$
\left.\begin{array}{c}(c_j',C_j')_{1,P_1}\,;\ldots;\,(c_j^{(p)},C_j^{(p)})_{1,P_p} \\ (d_j',D_j')_{1,Q_1}\,;\ldots;\,(d_j^{(p)},D_j^{(p)})_{1,Q_p}\end{array}\right] \qquad \ldots(3.1)
$$

(ii) **Further taking $a = 0$ in (2.1), we get**

$$
{}_cR^{\rho;\,\alpha,\,\beta,\,0;\,1,\,0,\,\mu}_{x;\,r,\,q;\,m,\,k,\,l}
$$

$$
\{x^{\lambda}(x^{v}+\xi)^{\sigma}\,S_{n}^{m_1,\ldots,m_s}\,[y_1x^{\lambda_1}(x^{v}+\xi)^{-\tau_1},\ldots,y_s\,x^{\lambda_s}(x^{v}+\xi)^{-\tau_s}]
$$

$$
H[z_1\,x^{\rho_1}(x^{v}+\xi)^{-\sigma_1},\ldots,z_p\,x^{\rho_p}(x^{v}+\xi)^{-\sigma_p}]\}
$$

$$
=\frac{1}{\Gamma(\mu)}\sum_{v=0}^{m+n}\sum_{u=0}^{v}\sum_{k_1,\ldots,k_s=0}^{m_1k_1+\ldots+m_sk_s\le n'}\sum_{w=0}^{\infty}\sum_{\delta=0}^{\infty}\frac{(-1)^{w+\delta}}{w!\,\delta!\,(\mu+A+\delta)}
$$

$$B(m, n, u, v)\, D(n', k_i, y_i)\, \xi^{\sigma - \sum_{i=1}^{s} \tau_i k_i - w} (x-c)^{\mu + A + \delta)}\, x^{\lambda + vw - \delta - A + \sum_{i=1}^{s} \tau_i k_i}$$

$$H^{0, N+2 : M_1, N_1 ;...; M_p, N_p}_{P+2, Q+2 : P_1, Q_1 ;...; P_p, Q_p} \left[\begin{array}{c} z_1 x^{\rho_1} \xi^{-\sigma_1} \\ \vdots \\ z_p x^{\rho_p} \xi^{-\sigma_p} \end{array} \right|$$

$$(-\lambda - vw - \sum_{i=1}^{s} \lambda_i k_i ; \rho_1, ..., \rho_p),$$

$$(b_j ; B_j' , ..., B_j^{(p)})_{1, Q},$$

$$(1 - w + \sigma - \sum_{i=1}^{s} \tau_i k_i ;\ \sigma_1, ..., \sigma_p), (a_j ; A_j', ..., A_j^{(p)})_{1, p} :$$

$$(\delta - \lambda - vw - \sum_{i=1}^{s} \lambda_i k_i ; \rho_1, ..., \rho_p), (1 + \sigma - \sum_{i=1}^{s} \tau_i k_i ;\ \sigma_1, ..., \sigma_p);$$

$$\left. \begin{array}{l} (c_j', C_j')_{1, P_1} ;...; (c_j^{(p)}, C_j^{(p)})_{1, P_p} \\ (d_j', D_j')_{1, Q_1} ;...; (d_j^{(p)}, D_j^{(p)})_{1, Q_p} \end{array} \right] \qquad ...(3.2)$$

(*iii*) Again taking $c = 0$ in (3.1), we find that

$$R^{\rho;\, \alpha, \beta, 0; 1, 0, \mu}_{x; r, q; m, k, l} \{x^{\lambda} (x^{v} + \xi)^{\sigma}$$

$$S^{m_1, ..., m_s}_{n'} [y_1 x^{\lambda_1} (x^{v} + \xi)^{-\tau_1}, ..., y_s x^{\lambda s} (x^{v} + \xi)^{-\tau_s}]$$

$$H[z_1 x^{\rho_1} (x^{v} + \xi)^{-\sigma_1}, ..., z_p x^{\rho_p} (x^{v} + \xi)^{-\sigma_p}]\}$$

$$= \frac{1}{\Gamma(\mu)} \sum_{v=0}^{m+n} \sum_{u=0}^{v} \sum_{k_1, ..., k_s = 0}^{m_1 k_1 + ... + m_s k_s \leq n'} \sum_{w=0}^{\infty} \frac{(-1)^{w}}{w!}$$

$$\xi^{\sigma - \omega - \sum_{i=1}^{\sigma} \tau_i \kappa_i}\, x^{\mu + \lambda + v + w + \sum_{i=1}^{s} \lambda_i k_i}\, \Gamma(\mu + A)\, B(m, n, u, v)\, D(n', k_i, y_i)$$

$$H^{0, N+2 : M_1, N_1 ;...; M_p, N_p}_{P+2, Q+2 : P_1, Q_1 ;...; P_p, Q_p} \left[\begin{array}{c} z_1 x^{\rho_1} \xi^{-\sigma_1} \\ \vdots \\ z_p x^{\rho_p} \xi^{-\sigma_p} \end{array} \right|$$

$$(-\lambda - vw - \sum_{i=1}^{s} \lambda_i k_i ;\ \rho_1, ..., \rho_p)$$

$$(b_j' ; B_j' , ..., B_j^{(p)})_{1, Q},$$

$$(1-w+\sigma-\sum_{i=1}^{s}\tau_i k_i\,;\sigma_1,\ldots,\sigma_p),(a_j\,;A_j',\ldots,A_j^{(p)})_{1,P}:$$

$$(-\mu-A-\lambda-vw-\sum_{i=1}^{s}\lambda_i k_i\,;\rho_1,\ldots,\rho_p),(1+\sigma-\sum_{i=1}^{s}\tau_i k_i\,;\sigma_1,\ldots,\sigma_p):$$

$$\left.\begin{matrix}(c_j',C_j')_{1,P_1};\ldots;(c_j^{(p)},C_j^{(p)})_{1,P_p}\\(d_j',D_j')_{1,Q_1};\ldots;(d_j^{(p)},D_j^{(p)})_{1,Q_p}\end{matrix}\right] \qquad \ldots(3.3)$$

(*iv*) If we take $n'=0$ and $\lambda_i, T_i \to 0$ $(i=l,\ldots,s)$ in (3.3), we get

$$R_{x;\,r,\,q;\,m,\,k,\,l}^{\rho;\,\alpha,\,\beta,\,0;\,1,\,0,\,\mu}\{x^{\lambda}(x^{v}+\xi)^{\sigma}$$

$$H[z_1 x^{\rho_1}(x^{v}+\xi)^{-\sigma_1},\ldots,z_p x^{\rho_p}(x^{v}+\xi)^{-\sigma_p}]\}$$

$$=\frac{1}{\Gamma(\mu)}\sum_{v=0}^{m+n}\sum_{u=0}^{v}\sum_{w=0}^{\infty}\frac{(-1)^{w}}{w!}B(m,n,u,v)\,\xi^{\sigma-\omega}x^{\mu+\lambda+vw}\Gamma(\mu+A)$$

$$H_{P+2,\,Q+2,\,:\,P_1,\,Q_1;\ldots;P_p,\,Q_p}^{0,\,N+2\,:\,M_1,\,N_1;\ldots;M_p,\,N_p}\left[\begin{matrix}z_1 x^{\rho_1}\xi^{-\sigma_1}\\ \vdots\\ z^{p}x^{\rho_p}\xi^{-\sigma_p}\end{matrix}\right|$$

$$(-\lambda-vw;\rho_1,\ldots,\rho_p),(1-w+\sigma;\sigma_1,\ldots,\sigma_p),(a_j;A_j',\ldots,A_j^{(p)})_{1,P}:$$

$$(b_j,B_j',\ldots,B_j^{(p)})_{1,Q},(-\mu-A-\lambda-vw;\rho_1,\ldots,\rho_p),(1+\sigma;\sigma_1,\ldots,\sigma_p):$$

$$\left.\begin{matrix}(c_j',C_j')_{1,P_1};\ldots;(c_j^{(p)},C_j^{(p)})_{1,P_p}\\(d_j',D_j')_{1,Q_1};\ldots;(d_j^{(p)},D_j^{(p)})_{1,Q_p}\end{matrix}\right] \qquad \ldots(3.4)$$

Further, if we put $m=n=\alpha=k=0$ in (3.4), we arrive at a recent result due to Srivastava et al. [9,p.563, Eq.(2.1)], which contains another known result due to Srivastava and Goyal [11,p.644, Eq.(2.1)] as a special case.

The conditions of validity of the fractional integral formula (3.1) through (3.4) are obtainable fairly easily from those of their parent formula (2.1).

(*v*) Now, appealing to the known relationship (c.f. [13, p.272, Eq.(4.7)], see also [12,p.253, Eq.(C.9)]), it is not difficult to derive the following fractional integral formula as a special case of (3.4):

$$R_{x;\,r,\,q;\,m,\,k,\,l}^{\rho;\,\alpha,\,\beta,\,0;\,1,\,0,\,\mu}\{x^{\lambda}(x^{v}+\xi)^{\sigma}$$

$$F^{P:P_1,\dots,P_p}_{Q:Q_1,\dots,Q_p}\left[\begin{array}{l}(a_j;A_j',\dots,A_j^{(p)})_{1,P}:\\(a_j;B_j',\dots,B_j^{(p)})_{1,Q}:\end{array}\right.$$

$$\left.\begin{array}{l}(c_j',C_j')_{1,P_1};\dots;(c_j^{(p)},C_j^{(p)})_{1,P_p};\\(d_j',D_j')_{1,Q_1};\dots;(d_j^{(p)},D_j^{(p)})_{1,Q_p};\end{array} z_1x^{\rho_1}(x^v+\xi)^{-\sigma_1},\dots,z_px^{\rho_p}(x^v+\xi)^{-\sigma_p}\right]\}$$

$$=\xi^{\sigma}x^{\mu+\lambda}\frac{\Gamma(\lambda+1)}{\Gamma(\mu)}\sum_{v=0}^{m+n}\sum_{u=0}^{v}B(m,n,u,v)\frac{\Gamma(\mu+A)}{\Gamma(1+\lambda+\mu+A)}$$

$$F^{P+2:P_1;\dots;P_p;0}_{Q+2:Q_1;\dots;Q_p;0}\left[\begin{array}{l}(1+\lambda;\rho_1,\dots,\rho_p,v),(-\sigma;\sigma_1,\dots,\sigma_p,1),\\(1+\lambda+\mu+A;\rho_1,\dots,\rho_p,v),(-\sigma;\sigma_1,\dots,\sigma_p,0),\end{array}\right.$$

$$(a_j;A_j',\dots,A_j^{(p)})_{1,P}:(c_j',C_j')_{1,P_1};\dots;(c_j^{(p)},C_j^{(p)})_{1,P_p}:$$

$$(b_j;B_j',\dots,B_j^{(p)})_{1,Q}:(d_j',D_j')_{1,Q_1};\dots;(d_j^{(p)},D_j^{(p)})_{1,Q_p}:$$

$$\left.z_1x^{\rho_1}\xi^{-\sigma_1},\dots,z_px^{\rho_p}\xi^{-\sigma_p},-\frac{x^v}{\xi}\right] \quad \dots(3.5)$$

(*vi*) Further, putting $\sigma=0$, $v=\rho_1=\dots=\rho_p=1$ and $\sigma_i\to 0$ $(i=1,\dots,p)$ in (3.5), we get

$$R^{\rho;\,\alpha,\,\beta,\,0;\,1,\,0,\,\mu}_{x;\,r,\,q;\,m,\,k,\,l}\left[x^{\lambda}F^{P:P_1;\dots;P_p}_{Q:Q_1;\dots;Q_p}\left|\begin{array}{l}(a_j;A_j',\dots,A_j^{(p)})_{1,P}:\\(b_j;B_j',\dots,B_j^{(p)})_{1,Q}:\end{array}\right.\right.$$

$$\left.\begin{array}{l}(c_j',C_j')_{1,P_1};\dots;(c_j^{(p)},C_j^{(p)})_{1,P_p};\\(d_j',D_j')_{1,Q_1};\dots;(d_j^{(p)},D_j^{(p)})_{1,Q_p};\end{array} z_1x,\dots,z_px\right]$$

$$=\frac{x^{\mu+\lambda}\Gamma(\lambda+1)}{\Gamma(\mu)}\sum_{v=0}^{m+n}\sum_{u=0}^{v}B(m,n,u,v)\frac{\Gamma(\mu+A)}{\Gamma(1+\lambda+\mu+A)}$$

$$F^{P+1:P_1;\dots;P_p}_{Q+1:Q_1;\dots;Q_p}\left[\begin{array}{l}(1+\lambda;1,\dots,1):\\(1+\lambda+\mu+A;1,\dots,1):\end{array}\right.$$

$$\left.\begin{array}{l}(c_j',C_j')_{1,P_1};\dots;(c_j^{(p)},C_j^{(p)})_{1,P_p};\\(d_j',D_j')_{1,Q_1};\dots;(d_j^{(p)},D_j^{(p)})_{1,Q_p};\end{array} z_1x,\dots,z_px\right] \quad \dots(3.6)$$

(*vii*) From (3.6), we obtain easily the following formulas involving Lauricella hypergeometric functions of p variables :

$$R^{\rho;\,\alpha,\,\beta,\,0;\,1,\,0,\,\mu}_{x;\,r,\,q;\,m,\,k,\,l}\{x^{\lambda}[a_1,\dots,a_p,b_1,\dots,b_p;1+\lambda;z_1x,\dots,z_px]\}$$

$$= \frac{x^{\mu+\lambda}\,\Gamma(\lambda+1)}{\Gamma(\mu)} \sum_{v=0}^{m+n} \sum_{u=0}^{v} B(m,n,u,v) \frac{\Gamma(\mu+A)}{\Gamma(1+\lambda+\mu+A)}$$

$$F_B^{(p)}\,[a_1,\dots,a_p, b_1,\dots,b_p\,;\,1+\lambda+\mu+A; z_1x,\dots,z_px]\} \qquad \text{...(3.7)}$$

$$R^{\rho;\,\alpha,\,\beta,\,0;\,1,\,0,\,\mu}_{x;\,r,\,q,\,m,\,k,\,l}\,\{x^{\lambda}\prod_{i=1}^{p}(1-z_ix)^{-\mu_i}\}$$

$$= \frac{x^{\mu+\lambda}\,\Gamma(\lambda+1)}{\Gamma(\mu)} \sum_{v=0}^{m+n} \sum_{u=0}^{v} B(m,n,u,v) \frac{\Gamma(\mu+A)}{\Gamma(1+\lambda+\mu+A)}$$

$$F_D^{(p)}\,[1+\lambda, b_1,\dots,b_p\,;\,1+\lambda+\mu+A; z_1x,\dots,z_px] \qquad \text{...(3.8)}$$

If in (3.8), we put $\rho = m = n = v = \alpha = k = 0$ and replace λ by $\lambda - 1$, μ by $\mu - \lambda$, we arrive at a result due to Srivastava and Goyal [11,p.649, Eq. (3.6)] which for $p = 2$ reduces to another known result due to Lavoie et al. [5,p.260].

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