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FUNCTIONS OF CLASS $Lip\,(\alpha, p)$ AND THEIR $F(a, q)$ MEAN

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ABSTRACT

In the present paper some results of Mohapatra, Holland and Sahney [6] on Taylor mean are extended to general class of $F(a, q)$ transform.

1. INTRODUCTION

Let $f \in L\,[-\pi, \pi]$ and be periodic with period 2π outside this range. Let the Fourier series associated with $f(x)$ be given by

$$f(x) \sim \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \qquad \text{... (1.1)}$$

We write

$$\varphi_x(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\} \qquad \text{... (1.2)}$$

The family $F(a, q)$ of summability methods was introduced by Meir[5]. The summability matrix $\{c_{pk}\}$ belongs to $F(a, q)$ if it satisfies the following conditions :

p is a discrete or continuous parameter; $q = q(p)$ is a positive increasing function which tends to infinity as $p \to \infty$, 'a' is a positive constant; for every fixed $\delta : \frac{1}{2} < \delta < \frac{2}{3}$.

$$c_p k = \sqrt{\frac{a}{\pi q}} \exp\left(-aq^{-1}(k-q)^2\right)\left\{1 + O\left(\frac{|k-q|+1}{q}\right) + O\left(\frac{|k-q|^3}{q^2}\right)\right\} \qquad \text{... (1.3)}$$

as $p \to \infty$ uniformly in k for $|k-q| \le q^{\delta}$; and

$$\sum_{|k-q| > q^{\delta}} (k+1)\, c_p k = O\{\exp(-q^{\mu})\} \qquad \text{... (1.4)}$$

where μ is some positive number independent of p.

Let

$$l_p(f;x) = \sum_{k=0}^{\infty} c_p k\, s_k(f;x) \qquad \dots (1.5)$$

denote the $F(a,q)$ mean of the Fourier series (1.1) of f, where $s_k(f;x)$ is the kth partial sum of (1.1).

The family $F(a,q)$ contains the summability methods of generalised Borel, Euler, Taylor, S_α and Valiron.

It is known (see Kuttner, Rajagopal and Rangachari[4]) that

$$\sum_{k=0}^{\infty} c_p k = 1 + O(q^{-1/2}) \qquad \dots (1.6)$$

The summability methods of Euler, Taylor, S_α and Borel satisfy (1.6) in the stronger form

$$\sum_{k=0}^{\infty} c_p k = 1 \qquad \dots (1.7)$$

Norms considered in this chapter are Lp ($p \geq 1$) norms and throughout the paper norms are taken with respect to variable x. The classes $Lip\ \alpha$, $Lip\ (\alpha, p)$ ($p \geq 1$) are as defined by Hardy and Littlewood [1, p.612] and Zygmund [7], p. 42, 45. The class $Lip\ (\alpha, p)$ reduces to $Lip\ \alpha$ with $p = \infty$.

For $p \geq 1$, $\delta > 0$, let $\omega(\delta, g)$ and $\omega_p(\delta, g)$ denote the modulus of continuity and integral modulus of continuity respectively of an appropriate function g.

2. In 1985 Mohapatra, Holland and Sahney [6] studied order of approximation by Taylor mean $T_n^r(f;x)$ of the Fourier series of $f(x)$ when $f(x)$ belongs to the class $Lip\ (\alpha, p)$ and proved the following theorems :

THEOREM A. If $f \in Lip\ (\alpha, p)$, $0 < \alpha \leq 1$, $p > 1$, then

$$\| T_n^r(f;x) - f(x) \| = O(n^{-\alpha\beta}) \qquad \dots (2.1)$$

for $0 < \beta < 1/2$.

THEOREM B. If $f \in Lp$, $(p > 1)$, then for $0 < \beta < 1$

$$\|T_n^r(f;x) - f(x)\| = O(\omega_p(1/n;f)) + O\left(\int_{a_n}^{(a_n)^\beta} \frac{\omega_p(t;f)}{t}\,dt\right)$$
$$+ O(n^\beta \exp(-An^{1-2\beta})) \quad \ldots (2.2)$$

where $a_n = \pi\left(n + \frac{1}{2} + \frac{n+1}{1-r}r\right)^{-1}$.

THEOREM C. If $f \in Lip\ \alpha$, $0 < \alpha \le 1, p > 1, \alpha p > 1$

$$T_n^r(f;x) - f(x) = O(n^{-\alpha - 1/p)\beta})\ (0 < \beta < 1/2) \quad \ldots (2.3)$$

uniformly in x almost everywhere.

Since $F(a, q)$ method includes Taylor method, it is natural to ask as to what be the result if we apply $F(a, q)$ mean in place of Taylor mean in the above theorems. In answer to this question our aim in the present paper is to extend the above results to the family $F(a, q)$.

We prove the following results :

THEOREM 1. Let $[q]$ denote the integral part of $q = q(p)$ and $m = [q] + 1$. If $f \in Lp$, $(p > 1)$ and

$$\frac{\omega_p(t, f)}{t^{1/2}} \text{ is decreasing function of } t \text{ in } 0 \le t \le \pi \quad \ldots (2.4)$$

then, for $0 < \beta < 1$,

$$\|t_p(f;x) - f(x)\| = O(\omega_p^{(1/m, f)}) + O\left(\int_{\pi/m}^{\pi/m^\beta} \frac{\omega_p(t;f)}{t}\,dt\right)$$
$$+ O(m^\beta \exp(-Am^{1-2\beta})) \quad \ldots (2.5)$$

where $A = \frac{\pi^2}{4a}$, a constant.

THEOREM 2. If $f \in Lip\ (\alpha, p)$, $0 < \alpha \le 1, p > 1$, then

$$\|t_p(f;x) - f(x)\| = O(m^{-\alpha\beta}) \quad \ldots (2.6)$$

for $0 < \beta < 1/2$, where m is as in the Theorem 1.

THEOREM 3. If $f \in Lip\ (\alpha, p)$, $0 < \alpha \le 1, p > 1, \alpha p > 1$

$$t_p(f;x) - f(x) = O(m^{-(\alpha - 1/p)\beta})\ (0 < \beta < 1/2) \quad \ldots (2.7)$$

uniformly in x almost everywhere.

3. We shall need the following lemmas :

LEMMA 1. If $h(x, t)$ is a function of two variables defined for $0 \le t \le \pi$, $0 \le x \le 2\pi$, then

$$\left\| \int h(x, t)\, dt \right\| \le \int \| h(x, t)\, dt \|_p \, dt \quad (p > 1).$$

This is due to Hardy, Littlewood and Polya[2].

LEMMA 2. If $q = q(p)$ is an integer valued function of p, then for $\frac{1}{2} < \delta < \frac{2}{3}$, we have

$$\int_0^\pi \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-q| \le q^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-1)^2) \sin(k+1/2)\, t\, dt$$

$$= \int_0^\pi \frac{\varphi_x(t)}{\sin t/2} \exp\left(-\frac{qt^2}{4a}\right) \sin(q+1/2)\, t\, dt + O(q \exp(-aq^{2\delta-1}))$$

PROOF OF LEMMA 2. From the proof of Lemma 3.2 of Ikeno[3], we have

$$\sum_{|k-q| \le q^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)\, t$$

$$= \exp\left(-\frac{qt^2}{4a}\right) \sin(q+1/2)\, t\, dt + O(q \exp(-aq^{2\delta-1})\, |t|)$$

Hence the Lemma 2 follows in view of boundedness of $\varphi_x(t)$ and since $\sin t/2 > t/\pi$ $(0 < t < \pi)$.

LEMMA 3. If m is as defined in the Theorem 1, we have

$$\int_0^\pi \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-q| \le q^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)\, t\, dt$$

$$= \int_0^\pi \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-m| \le m^\delta} \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m)^2) \sin(k+1/2) t\, dt + O(\omega_p(1/m, f))$$

PROOF OF LEMMA 3. We estimate the difference

$$\int_0^\pi \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-q| \le q^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2) t\, dt$$

$$- \int_0^\pi \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-m| \le m^\delta} \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m)^2) \sin(k+1/2) t\, dt$$

$$= \int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{m \le k \le m+m^{\delta}} \left[\sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \right.$$

$$\left. - \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m)^2) \right] \sin(k+1/2)\, t\, dt$$

$$+ \int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{m - m^{\delta} \le k < m} \left[\sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \right.$$

$$\left. - \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m)^2) \right] \sin(k+1/2)t\, dt$$

$$- \int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{q+q^{\delta} < k \le m+m^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)t\, dt$$

$$+ \int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{q-q^{\delta} \le k \le m-m^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)t\, dt$$

$$= D_1 + D_2 + D_3 + D_4 \qquad \text{... (3.1)}$$

From Ikeno[3], p.261, 262, we have

$$\sum_{q+q^{\delta} < k \le m+m^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)t = O(\sqrt{q} \exp(-aq^{2\delta-1}))$$

and also

$$\sum_{q-q^{\delta} < k \le m-m^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)t = O(\sqrt{q} \exp(-aq^{2\delta-1}))$$

Thus, using Lemma 1

$$\| D_3 \| \le \int_0^{\pi} \frac{\| \varphi_x(t) \|}{t} \sum_{q+q^{\delta} < k \le m+m^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)\, |\sin(k+\tfrac{1}{2})t|\, dt$$

$$= O(\sqrt{q} \exp(-aq^{2\delta-1})). \qquad \text{... (3.2)}$$

Similarly,

$$\| D_4 \| = O(\sqrt{q} \exp(-aq^{2\delta-1})) \qquad \text{... (3.3)}$$

In case where $q < m \le k \le m + m^{\delta}$, we have

$$O \le \frac{(k-m)}{\sqrt{m}} < \frac{(k-q)}{\sqrt{q}} < \frac{(k-[q])}{\sqrt{[q]}}.$$

Therefore, from Ikeno [3, p.250]

$$\left| \sqrt{\frac{a}{\pi q}} \exp\left(-aq^{-1}(k-q)^2\right) - \sqrt{\frac{a}{\pi m}} \exp\left(-am^{-1}(k-m)^2\right) \right|$$

$$= O\left\{ \frac{1}{\sqrt{m}} \exp\left(-am^{-1}(k-m)^2\right) \left(\frac{(k-m)^2}{m^2} + \frac{|k-m|}{m} + \frac{1}{m} \right) \right\} \qquad \text{... (3.4)}$$

Using (3.3) and Lemma 1, we obtain

$$\|D_1\| = O\left[\int_0^{\pi} \frac{\|\varphi_x(t)\|}{t} \sum_{m \le k \le m+m^{\delta}} \left\{ \frac{1}{\sqrt{m}} \exp\left(-am^{-1}(k-m)^2\right) \left(\frac{(k-m)^2}{m^2} + \frac{|k-m|}{m} + \frac{1}{m} \right) \right\} |\sin(k+½)t| \, dt \right]$$

$$= O\left[\int_0^{\pi/m} \frac{\|\varphi_x(t)\|}{t} \sum_{m \le k \le m+m^{\delta}} \left\{ \frac{1}{\sqrt{m}} \exp\left(-am^{-1}(k-m)^2\right) \left(\frac{(k-m)^2}{m^2} + \frac{|k-m|}{m} + \frac{1}{m} \right) \right\} (|k-m| + m + ½)t \, dt \right]$$

$$+ O\left[\int_{\pi/m}^{\pi} \frac{\|\varphi_x(t)\|}{t} \sum_{m \le k \le m+m^{\delta}} \left\{ \frac{1}{\sqrt{m}} \exp\left(-am^{-1}(k-m)^2\right) \left(\frac{(k-m)^2}{m^2} + \frac{|k-m|}{m} + \frac{1}{m} \right) \right\} dt \right]$$

$$= O\left\{ \sqrt{m} \int_0^{\pi/m} \|\varphi_x(t)\| \, dt \right\} + O\left\{ \frac{1}{\sqrt{m}} \int_{\pi/m}^{\pi} \frac{\|\varphi_x(t)\|}{t} \, dt \right\}$$

$$= O\left\{ \sqrt{m} \int_0^{\pi/m} \omega_p(t, f) \, dt \right\} + O\left\{ \frac{1}{\sqrt{m}} \int_{\pi/m}^{\pi} \frac{\omega_p(t, f)}{t} \, dt \right\}$$

$$= O\left(\omega_p(1/m.\, f)\right) \qquad \text{... (3.5)}$$

Also in case where $k \le [q] \le q < m$, we get

$$\frac{(k-m)}{\sqrt{m}} < \frac{(k-q)}{\sqrt{q}} < \frac{(k-[q])}{\sqrt{[q]}} \le 0.$$

Hence from Ikeno [3, p.260] the following estimate results :

$$\left| \sqrt{\frac{a}{\pi q}} \exp\left(-aq^{-1}(k-q)^2\right) - \sqrt{\frac{a}{\pi m}} \exp\left(-am^{-1}(k-m)^2\right) \right|$$

$$= O\left(\frac{1}{\sqrt{[q]}} \exp\left(-a[q]^{-1}(k-[q])^2\right) \left(\frac{(k-[q])^2}{[q]^2} + \frac{|k-[q]|}{[q]} + \frac{1}{[q]} \right)\right)$$

Proceeding as in the estimation of D_1 and using the above inequality we get

$$\| D_2 \| = \omega_p (1/m, f). \qquad \ldots (3.6)$$

Thus, Lemma 3 follows from (3.1) to (3.6).

LEMMA 4. Suppose that $f \in Lip\ (\alpha, p)$, where $p > 1$, $0 < \alpha \le 1$, $\alpha p > 1$. Then f is equal to a function $g \in Lip\ (\alpha - 1/p)$ almost everywhere.

This is due to Hardy and Littlewood[1], Theorem 5(ii), p.627.

4. PROOF OF THEOREM 1. Since

$$s_k(f; x) - f(x) = \frac{1}{\pi} \int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sin (k + 1/2)t\, dt$$

we have

$$t_p(f; x) - f(x) = \frac{1}{\pi} \int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{k=0}^{\infty} c_p\, k \sin (k + 1/2)t\, dt + O(q^{-1/2})$$

$$= \frac{1}{\pi} \int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \left[\left(\sum_{|k-q| \le q^{\delta}} + \sum_{|k-q| > q^{\delta}} \right) c_p\, k \sin (k + 1/2)t \right] dt + O(q^{-1/2})$$

$$= S_1 + S_2 + O(q^{-1/2}). \qquad \ldots (4.1)$$

By the Minkowski Inequality

$$\| t_p(f; x) - f(x) \| \le \| S_1 \| + \| S_2 \| + O(q^{-1/2}) \qquad \ldots (4.2)$$

By (1.4), Lemma 1 and the fact that $\sin t/2 \ge t/\pi$ $(O < t \le \pi)$, we obtain

$$\| S_2 \| \le \int_0^{\pi} \frac{\| \varphi_x(t) \|}{t} \sum_{|k-q| > q^{\delta}} c_p\, k\, (k + 1/2)t\, dt = O(\exp(-q^{\mu})). \qquad \ldots (4.3)$$

Now making use of (1.3), we can write

$$S_1 = \frac{1}{\pi}\int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-q|\le q^{\delta}} \sqrt{\frac{a}{\pi q}}\ \exp(-aq^{-1}(k-q)^2)$$

$$\left\{1 + O\left(\frac{|k-q|+1}{q}\right) + O\left(\frac{|k-q|^3}{q^2}\right)\right\} \sin(k+1/2)t\, dt$$

$$= S_3 + S_4 + S_5. \qquad \text{... (4.4)}$$

Using Lemma 1, we estimate S_4 as follows :

$$\| S_4 \| \le \int_0^{\pi} \frac{\|\varphi_x(t)\|}{t} \sum_{|k-q|\le q^{\delta}} \left\{\sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)\, O\left(\frac{|k-q|+1}{q}\right)\right\}$$

$$|\sin(k+½)t|\, dt$$

$$= \int_0^{\pi/q} \frac{\|\varphi_x(t)\|}{t} \sum_{|k-q|\le q^{\delta}} \left\{\sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)\, O\left(\frac{|k-q|+1}{q}\right)\right\}$$

$$(|k-q| + q + ½)t\, dt$$

$$+ \int_{\pi/q}^{\pi} \frac{\|\varphi_x(t)\|}{t} \sum_{|k-q|\le q^{\delta}} \sqrt{\frac{a}{\pi q}}\ \exp(-aq^{-1}(k-q)^2)\, O\left(\frac{|k-q|+1}{q}\right) dt$$

$$= O\left(\sqrt{q}\int_0^{\pi/q} \|\varphi_x(t)\|\, dt\right) + O\left(\frac{1}{\sqrt{q}}\int_{\pi/q}^{\pi} \frac{\|\varphi_x(t)\|}{t}\, dt\right)$$

$$= IO(\omega_p(1/q, f)). \qquad \text{... (4.5)}$$

Similarly,

$$\| S_5 \| = O(\omega_p(1/q, f)). \qquad \text{... (4.6)}$$

Applying Lemma 2 and Lemma 3 and noting that $m = m(p)$ is an integer valued function of p we get

$$S_3 = \int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \exp\left(-\frac{mt^2}{4a}\right) \sin(m+½)t\, dt + O(\omega_p(1/m, f))$$

$$\text{... (4.7)}$$

Now we write

$$I_1 = \left(\int_0^{\pi/m} + \int_{\pi/m}^{\pi/m\,\beta} + \int_{\pi/m\,\beta}^{\pi}\right) \frac{\varphi_x(t)}{\sin t/2} \exp\left(-\frac{mt^2}{4a}\right) \sin(m+½)t\, dt$$

$$= I_{1.2} + I_{1.2} + I_{1.3} \qquad \text{... (4.8)}$$

Since $\sin t/2 \leq t/\pi$ $(0 < t \leq \pi)$, we have by Lemma 1

$$\|I_{1.1}\| \leq \pi \int_0^{\pi/m} \frac{\|\varphi_x(t)\|}{t} \exp\left(-\frac{mt^2}{4a}\right) \sin(m + 1/2)t\, dt$$
$$\leq O\left(\int_0^{\pi/m} \frac{\omega_p(t, f)}{t}(m + 1/2)t\, dt\right)$$
$$= O(\omega_p(1/m, f)). \qquad \text{... (4.9)}$$

Further, by Lemma 1

$$\|I_{1.3}\| \leq O\left(\int_{\pi/m^{\beta}}^{\pi} \frac{\omega_p(t, f)}{t} \exp\left(-\frac{mt^2}{4a}\right) dt\right)$$
$$= O\{m^{\beta} \exp(-Am^{1-2\beta})\}. \qquad \text{... (4.10)}$$

Finally, by using Lemma 1 and the fact that $|\sin x| \leq 1$ for all x, we obtain

$$\|I_{1.2}\| \leq \int_{\pi/m}^{\pi/m^{\beta}} \frac{\|\varphi_x(t)\|}{t} \exp\left(-\frac{mt^2}{4a}\right) dt$$
$$= \int_{\pi/m}^{\pi/m^{\beta}} \frac{\omega_p(t, f)}{t} \exp\left(-\frac{mt^2}{4a}\right) dt$$
$$= O\left(\int_{\pi/m}^{\pi/m^{\beta}} \frac{\omega_p(t, f)}{t} dt\right). \qquad \text{... (4.11)}$$

Estimates (4.1), (4.2), ..., (4.11) togther yield the proof of the Theorem 1.

PROOF OF THE THEOREM 2. For $\beta < \frac{1}{2}$, we have

$$m^{\beta} \exp(-Am^{1-2\beta}) = \omega_p(1/m, f). \qquad \text{... (4.12)}$$

Since $f \in Lip(\alpha, p)$, $0 < \alpha \leq 1$, $p > 1$,

$$\omega_p(1/m, f) = O(m^{-\alpha}). \qquad \text{... (4.13)}$$

Now to deduce the Theorem 2 from Theorem 1 it suffices to show that

$$\int_{\pi/m}^{\pi/m^{\beta}} \frac{\omega_p(t, f)}{t} dt = O(m^{-\alpha\beta}). \qquad \text{... (4.14)}$$

Above equation follows from the fact that

$$\int_{\pi/m}^{\pi/m} t^{\alpha-1} dt = O(m^{-\alpha\beta}) + O(m^{-\alpha}).$$

PROOF OF THE THEOREM 3. Since $f \in Lip(\alpha, p)$, $0 \le \alpha < 1$, $p > 1$, $\alpha p > 1$ from Lemma 4 we see that there exists a function $g \in Lip(\alpha - 1/p)$, which is equal to f almost everywhere. Hence, we have almost everywhere

$$\varphi_x(t) = O(t^{\alpha - 1/p}). \qquad \dots (4.15)$$

Now in order to prove the Theorem 3 we modify estimates of D_1, D_2, D_3, D_4 in the proof of Lemma 3.

Using arguments similar to those used in estimating D_3

$$D_3 = O\left[\int_0^\pi \frac{|\varphi_x(t)|}{t} \sum_{q+q^\delta < k \le m+m^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \,|\sin(k+1/2)t|\, dt\right]$$

$$= O(\sqrt{q}\exp(-aq^{2\delta-1})). \qquad \dots (4.16)$$

Similarly,

$$D_4 = O(\sqrt{q}\exp(-aq^{2\delta-1})). \qquad \dots (4.17)$$

As in the proof of Lemma 3, we get

$$D_1 = O\left\{\sqrt{m}\int_0^{\pi/m} |\varphi_x(t)|\, dt\right\} + O\left\{\frac{1}{\sqrt{m}}\int_{\pi/m}^{\pi} \frac{|\varphi_x(t)|}{t}\, dt\right\}$$

$$= O(m^{-\alpha + 1/p}). \qquad \dots (4.18)$$

In a similar manner, we can obtain

$$D_2 = O(m^{-\alpha + 1/p}). \qquad \dots (4.19)$$

Thus, combining (3.1), (4.16), (4.17) and (4.19), we have

$$\int_0^\pi \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-q| \le q^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)\sin(k+1/2)t\, dt$$

$$= \int_0^\pi \frac{\varphi_x(t)}{\sin t/2} \sum_{|k-m| \le m^\delta} \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m)^2)\sin(k+1/2)t\, dt$$

$$+ O(m^{-\alpha + 1/p}) \qquad \dots (4.20)$$

In the notations of the Theorem 1.

$$t_p(f; x) - f(x) \le S_1 + S_2 + O(q^{-1/2}). \qquad \dots (4.21)$$

Using (1.4), we get

$$S_2 = O\left(\int_0^{\pi} \frac{|\varphi_x(t)|}{t} \sum_{|k-q|>q^{\delta}} c_p\, k\, (k + ½)t\, dt \right)$$

$$= O(\exp(-q^{\mu})). \qquad \ldots (4.22)$$

From (4.4), we have

$$S_1 = S_3 + S_4 + S_5. \qquad \ldots (4.23)$$

On the lines of the esimation of S_4 in the proof of the Theorem 1 and using (4.15), we obtain

$$S_4 = O\left(\sqrt{q} \int_0^{\pi/q} |\varphi_x(t)|\, dt \right) + O\left(\frac{1}{\sqrt{q}} \int_{\pi/q}^{\pi} \frac{|\varphi_x(t)|}{t}\, dt \right)$$

$$= O(q^{-\alpha + 1/p}). \qquad \ldots (4.24)$$

Similarly,

$$S_5 = O(q^{-(\alpha + 1/p)}). \qquad \ldots (4.25)$$

Applying Lemma 2 and (4.20) and using the fact that $m = m(p)$ is an integer valued function of p, we get

$$S_3 = \int_0^{\pi} \frac{\varphi_x(t)}{\sin t/2} \exp\left(-\frac{mt^2}{4a} \right) \sin (m + ½)t\, dt + O(m^{-(\alpha + 1/p)})$$

$$= I_1 + O(m^{-(\alpha + 1/p)}).$$

As in the Theorem 1, we write

$$I_1 = \left(\int_0^{\pi/m} + \int_{\pi/m}^{\pi/m\, \beta} + \int_{\pi/m\, \beta}^{\pi} \right) \frac{\varphi_x(t)}{\sin t/2} \exp\left(-\frac{mt^2}{4a} \right) \sin (m + ½)t\, dt$$

$$= I_{1.1} + I_{1.2} + I_{1.3}\,. \qquad \ldots (4.26)$$

Now using (4.15) we shall modify the estimates of $I_{1.1}$, $I_{1.2}$, $I_{1.3}$.

We have

$$I_{1.1} = O\left(\int_0^{\pi/m} \frac{|\varphi_x(t)|}{t} \exp\left(-\frac{mt^2}{4a} \right) \sin m + ½)t\, dt \right)$$

$$= O\left(\int_0^{\pi/m} \frac{t^{\alpha - 1/p}}{t} (m + ½)t\, dt \right)$$

$$= O(m^{-\alpha + 1/p}). \qquad \ldots (4.27)$$

Again,

$$I_{1.3} = O\left(\int_{\pi/m\,\beta}^{\pi} t^{\alpha - 1/p - 1} \exp\left(-\frac{mt^2}{4a}\right) dt\right)$$

$$= O\left(m^{-(\alpha - 1/p)\,\beta}\right). \qquad \text{... (4.28)}$$

Lastly,

$$I_{1.2} = O\left(\int_{\pi/m}^{\pi/m\,\beta} \frac{|\varphi_x(t)|}{t} \exp\left(-\frac{mt^2}{4a}\right) dt\right)$$

$$= O\left(\int_{\pi/m}^{\pi/m\,\beta} t^{\alpha - 1/p - 1}\, dt\right)$$

$$= O\,(m - (\alpha - 1/p)\beta) + O\,(m^{-\alpha + 1/p})$$

$$= O\left(m^{-(\alpha - 1/p)\,\beta}\right). \qquad \text{... (4.29)}$$

Hence the Theorem 3 follows from (4.21) to (4.29).

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