Jñānābha ,Vol. 27, 1997

STABILITY OF TWO RIVLIN-ERICKSEN ELASTICO-VISCOUS SUPERPOSE FLUIDS

By

R.C. Sharma and Pardeep Kumar
Department of Mathematics, Himachal Pradesh University
Summer Hill, Shimla-171005, India

(Received : September 12, 1996)

ABSTRACT

The stability of the plane interface separating two Rivilin-Ericksen elastico-viscous superposed fluids of uniform densities has been studied. The stability analysis has been carried out, for mathematical simplicity, for two highly viscous fluids of equal kinematic viscosities and equal kinematic viscoelasticities. It is found that the system is stable for stable configuration and unstable for unstable configuration. The behaviour of growth rates with respect to kinematic viscosity and kinematic viscoelasticity parameters are examined analytically.

1. INTRODUCTION The instability of the plane interface separating two fluids when one is accelerated towards the other or when one is superposed over the other has been studied by several authors, and Chandrasekhar [2] has given a detailed account of these investigations. Roberts [5] has extended the analysis to the case of two fluids of equal kinematic viscosities in presence of vertical magnetic field while Gerwin [3] has studied the case of compressible streaming fluids. The influence of viscosity on the stability of the plane interface separating two incompressible superposed fluids of uniform densities, when the whole system is immersed in a uniform horizontal magnetic field has been studied by Bhatia [1]. He has carried out the stability analysis for two fluids of equal kinematic viscosities and different uniform densities.

Sharma and Sharma [6] have studied the stability of the plane interface separating two viscoelastic (Oldroydian) superposed fluids of uniform densities in another study Sharma [7] has studied the instability of the plane interface between two Oldrydian viscoelastic superposed conducting fluids in the presence of a uniform magnetic field. With the growing importance of non-Newtonian fluids in modern technology and industries, the further investigations on such fluids are desirable. There are many non-Newtonian fluids that cannot be characterized by Oldroyd's [4] constitutive relations. Rivlin-Ericksen elastico- viscous fluid is one such fluid. It is this class of elastico-viscous fluids we are interested in, particularly to study the stability of the plane interface separating two imcompressible superposed Rivlin-Ericksen fluids of uniform densities. This aspect forms the subject matter of the present paper wherein we have carried out the stability analysis for two fluids of equal kinematic viscosities, equal kinematic viscoelasticities and different uniform densities.

2. PERTURBATION EQUATIONS Consider a static state, in which an incompressible Rivlin- Ericksen elastico-viscous fluid of variable density is arranged in horizontal strata and the pressure p and density ρ are functions of the vertical coordinate z only. The character of the equilibrium of this initial static state is determined, as usual, by supposing that the system is slightly disturbed and then by following its further evolution.

Let $\vec{v}(u, v, w)$, $\delta\rho$ and δp denote the perturbations in velocity (0,0,0), density ρ and pressure p, respectively. Then the linearized perturbation equations relevant to the problem are

$$\rho \frac{\partial \vec{v}}{\partial t} = -\nabla \delta p + \vec{g}\delta\rho + \rho \left(\nu + \nu' \frac{\partial}{\partial t}\right) \nabla^2 \vec{v} + \left(\frac{\partial \mu}{dz} + \frac{\partial}{\partial t}\frac{\partial \mu'}{dz}\right)\left(\frac{\partial w}{\partial \vec{x}} + \frac{\partial \vec{v}}{\partial z}\right) \quad \ldots(1)$$

$$\nabla \cdot \vec{v} = 0 \quad \ldots(2)$$

$$\frac{\partial}{\partial t}(\delta\rho) = -wD\rho, \quad \ldots(3)$$

where $\nu \left(= \frac{\mu}{\rho}\right)$ and $\nu' \left(= \frac{\mu'}{\rho}\right)$ denote respectively the kinematic viscosity and the kinematic viscoelasticity of the fluid, $\vec{g} = (0, 0, -g)$ is the acceleration due to gravity $\bar{x} = (x, y, z)$ and $D = \frac{d}{dz}$. Equation (3) ensures that the density of every particle remains unchanged as we follow it with its motion.

Analyzing the disturbances into normal modes, we assume that the perturbed quantities have the space and time dependence of the form

$$f(z) \exp\left(ik_x x + ik_y y + nt\right), \quad \ldots(4)$$

where k_x, k_y are the wave numbers along the x and y directions respectively, $k = (k_x^2 + k_y^2)^{1/2}$ is the resultant wave number, n is the growth rate which is, in general, a complex constant and $f(x)$ is some function of z.

For perturbations of the form (4), equations (1)-(3) become

$$\rho n u = -ik_x\,\delta p + \rho(\nu + \nu' n)(D^2 - k^2)u + (D\mu + nd\mu')(ik_x w + Du) \qquad ...(5)$$

$$\rho n v = -ik_y\delta p + \rho(\nu + \nu' n)(D^2 - k^2)v + (D\mu + nd\mu')(ik_y w + Dv), \qquad ...(6)$$

$$\rho n w = -D\delta p - g\delta\rho + \rho(\nu + \nu' n)(D^2 - k^2)w + 2(D\mu + nd\mu')dw \qquad ...(7)$$

$$ik_x u + ik_y v + Dw = 0 \qquad ...(8)$$

and

$$n\delta\rho = -wD\rho \qquad ...(9)$$

Multiplying equations (5) and (6) by $-ik_x$ and $-ik_y$, respectively, adding the resultant equations and using equation (8) in it, we obtain

$$\rho n Dw = -k^2\delta p + \rho(\nu + \nu' n)(D^2 - k^2)Dw + (D\mu + nD\mu')(D^2 + k^2)w \quad ...(10)$$

Substituting for $\delta\rho$ from (9) in equation (7) and eliminating δp between equations (7) and (10), we obtain the equation in w :

$$n[D(\rho Dw) - k^2\rho w] - [D\{\rho(v + v'n)(D^2 - k^2)Dw\}$$

$$- k^2\rho(\nu + \nu' n)(D^2 - k^2)w] + \frac{gk^2}{n}(D\rho)w$$

$$- [D\{(D\mu + nd\mu')(D^2 + k^2)w\} - 2k^2 D\,(\mu + n\mu')Dw] = 0 \qquad ...(11)$$

3. TWO SUPERPOSED RIVLIN-ERICKSEN FLUIDS SEPARATED BY A HORIZONTAL BOUNDARY

Here we consider the case when two Rivlin-Ericksen superposed fluids of uniform densities ρ_1, ρ_2 uniform viscosities μ_1, μ_2 and uniform viscoelasticities μ_1', μ_2' are separated by a horizontal boundary at z=0. The subscripts 1 and 2 distinguish the lower and the upper fluids, respectively. Then, in each region of constant ρ constant μ and constant μ', equation (11) becomes

$$(D^2 - k^2)(D^2 - q^2)w = 0 \qquad ...(12)$$

where

$$q^2 = k^2 + \frac{n}{\nu + \nu' n}.$$

Since w must vanish both when $z \to \infty$ (in the upper fluid) and $z \to -\infty$ (in the lower fluid), the general solution of equation (12) can be written as

$$w_1 = A_1 e^{+kz} + A_2 e^{+q_1 z}, \ (z < 0) \qquad \text{...(13)}$$

$$w_2 = A_3 e^{-kz} + A_4 e^{-q_2 z}, \ (z > 0) \qquad \text{...(14)}$$

where A_1, A_2, A_3, A_4 are constants of integration,

$$q_1 = \sqrt{k^2 + \frac{n}{\nu_1 + \nu_1' n}} \text{ and } q_2 = \sqrt{k^2 + \frac{n}{\nu_2 + \nu_2' n}}, \qquad \text{...(15)}$$

In writing the solutions (13) and (14), it is assumed that q_1 and q_2 are so defined that their real parts are positive.

4. BOUNDARY CONDITIONS The solutions (13) and (14) must satisfy certain boundary conditions. The boundary conditions to be satisfied at the interface $z = 0$ are :

$$w, \qquad \text{...(16)}$$

$$Dw, \qquad \text{...(17)}$$

and

$$(\mu + \mu' n)(d^2 + k^2) w, \qquad \text{...(18)}$$

must be continuous.

Integrating equation (11) across the interface $z = 0$, we obtain another condition

$$[\rho_2 Dw_2 - \rho_1 Dw_1]_{z=0} - [\tfrac{1}{n}(\mu_2 + \mu_2' n)(D^2 - k^2) Dw_2 - \tfrac{1}{n}(\mu_1 + \mu_1' n)(D^2 - k^2) Dw_1]_{z=0}$$

$$= -\frac{gk^2}{n^2}[\rho_2 - \rho_1] w_0 - \frac{2k^2}{n}(\mu_2 + n\mu_2' - \mu_1 - n\mu_1')(Dw)_0 \qquad \text{...(19)}$$

where w_0, $(Dw)_0$ are the common values of w_1, w_2 and Dw_1, Dw_2 at $z = 0$.

5. DISPERSION RELATION AND DISCUSSION Applying the boundary conditions (16)-(19) to the solutions (13) and (14), we obtain

$$A_1 + A_2 = A_3 + A_4 \ (= w_0) \qquad \text{...(20)}$$

$$kA_1 + q_1 A_2 = -kA_3 - q_2 A_4 \ (= Dw_0). \qquad \text{...(21)}$$

$$(\mu_1 + \mu_1' n)[2k^2 A_1 + (q_1^2 + k^2) A_2] = (\mu_2 + \mu_2' n)[2k^2 A_3 + (q_2^2 + k^2) A_4] \ (= \mu + \mu' n)(d^2 + k^2) w_0) \qquad \text{...(22)}$$

$$[-k\rho_2A_3-\rho_2q_2A_4-k\rho_1A_1-\rho_1A_1-\rho_1q_1A_2]-\frac{1}{n}(\mu_2-\mu_2'n)(-q_2)$$

$$(q_2^2-k^2)A_4-\frac{1}{n}(\mu_1+\mu_1'n)q_1(q_1^2-k^2)A_2]+\frac{gk^2}{2n^2}(\rho_2-\rho_1)$$

$$(A_1+A_2+A_3+A_4)+\frac{k^2}{n}(\mu_2+n\mu_2'-\mu_1-n\mu_1')$$

$$(kA_1+q_1A_2-kA_3-q_2A_4)=0 \qquad ...(23)$$

Eliminating the constant A_1, A_2, A_3, A_4 from equations (20)-(23), we obtain

$$\begin{vmatrix} 1 & 1 & -1 & -1 \\ k & q_1 & k & q_2 \\ 2k^2(\mu_1+\mu_1'n) & [(\mu_1+\mu_1'n)(q_1^2+k^2)] & -2k^2(\mu_2+\mu_2'n) & [-(\mu_2+\mu_2'n)(q_2^2+k^2)] \\ \begin{matrix}[-\alpha_1+\frac{R}{2}\\+\frac{k^2}{n}(\nu_2\alpha_2+\\+n\nu_2'\alpha_2-\\-\nu_1\alpha_1-n\nu_1'\alpha)]\end{matrix} & \begin{matrix}[\frac{R}{2}+\frac{k}{n}(\nu_2\alpha_2+\\+n\nu_2'\alpha_2-\upsilon_1\alpha_1-\\-n\nu_1'\alpha_1)q_1]\end{matrix} & \begin{matrix}[-\alpha_2+\frac{R}{2}\\-\frac{k^2}{n}(\nu_2\alpha_2+\\+n\nu_2'\alpha_2-\\-\nu_1\alpha_1-n\nu_1'\alpha_1)]\end{matrix} & \begin{matrix}[\frac{R}{2}-\frac{k}{n}(\nu_2\alpha_2+\\+n\nu_2'\alpha_2-\nu_1\alpha_1\\-n\nu_1'\alpha_1)q_2]\end{matrix} \end{vmatrix}=0$$

...(24)

where $\alpha_1=\dfrac{\rho_1}{\rho_1+\rho_2}$, $\alpha_2=\dfrac{\rho_2}{\rho_1+\rho_2}$, $R=\dfrac{gk}{n^2}(\alpha_2-\alpha_1)$.

The determinant can be reduced by substracting the first column from the second, the third column from the fourth and adding the first column to the third. By this procedure, we obtain

$$\begin{vmatrix} q_1-k & 2k & q_2-k \\ \begin{matrix}[(\nu_1\alpha_1+n\nu_1'\alpha_1)\\(q_1^2-k^2)]\end{matrix} & \begin{matrix}[-2k^2(\nu_2\alpha_2+n\nu_2'\alpha_2\\-\nu_1\alpha_1-n\nu_1'\alpha_1)]\end{matrix} & \begin{matrix}[-(\nu_2\alpha_2+n\nu_2'\alpha_2)\\(q_2^2-k^2)]\end{matrix} \\ \begin{matrix}[\frac{k}{n}(\nu_2\alpha_2+n\nu_2'\alpha_2\\-\nu_1\alpha_1-n\nu_1'\alpha_1)\\(q_1-k)+\alpha_1]\end{matrix} & R-1 & \begin{matrix}[-\frac{k}{n}(\nu_2\alpha_2+n\nu_2'\alpha_2\\-\nu_1\alpha_1-n\nu_1'\alpha_1)\\(q_2-k)+\alpha_2]\end{matrix} \end{vmatrix}=0$$

...(25)

Evaluating the determinant (25), we obtain the following characteristic equation

$$(q_1 - k)[-2k^2 (\nu_2\alpha_2 + n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1) \{-\tfrac{k}{n} (\nu_2\alpha_2$$
$$+ n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1) (q_2 - k) + \alpha_2\} + (R-1) (\nu_2\alpha_2$$
$$+ n\nu_2'\alpha_2) (q_2^2 - k^2)] - 2k [(\nu_1\alpha_1 + n\nu_1'\alpha_1)(q_1^2 - k^2)$$
$$\{-\tfrac{k}{n} (\nu_2\alpha_2 + n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1) (q_2 - k) + \alpha_2\} +$$
$$+ (\nu_2\alpha_2 + n\nu_2'\alpha_2) (q_2^2 - k^2) (\tfrac{k}{n} (\nu_2\alpha_2 + n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1)$$
$$(q_1 - k) + \alpha_1\}] + (q_2 - k) [(\nu_1\alpha_1 + n\nu_1'\alpha_1)(q_1^2 - k^2)(R-1)$$
$$+ 2k^2 (\nu_1\alpha_2 + n\nu_2'\alpha_2 - \nu_1\alpha_1 - n\nu_1'\alpha_1) \{ \tfrac{k}{n} (\nu_2\alpha_2 + n\nu_2'\alpha_2$$
$$- \nu_1\alpha_1 - n\nu_1'\alpha_1) (q_1 - k) + \alpha_1\}] = 0. \quad ...(26)$$

The dispersion relation (26) is quite complicated as the values of q_1 and q_2 involve square roots. We therefore make the assumption that the two fluids are of high viscosity and high viscoelasticity. Under this assumption, we have

$$q = k\left[1 + \frac{n}{k^2 (\nu + \nu' n)}\right]^{1/2} = k + \frac{n}{2k (\nu + \nu' n)}, \quad ...(27)$$

so that

$$q_1 - k = \frac{n}{2k (\nu_1 + \nu_1' n)} \text{ and } q_2 - k = \frac{n}{2k (\nu_2 + \nu_2' n)} \quad ...(28)$$

Substituting the values of $q_1 - k$ and $q_2 - k$ from (28) in equation (26) and putting $\nu_1 = \nu_2 = \nu$, $\nu_1' = \nu_2' = \nu'$ (the case of equal kinematic viscosities and equal kinematic viscoelasticities, for mathematical simplicity), we obtain the following dispersion relation

$$(1 + 2k^2\nu')n^2 + (2k^2\nu)n - gk(\alpha_2 - \alpha_1) = 0. \quad ...(29)$$

For the potentially stable arrangement $\alpha_1 > \alpha_2$ equation (29) does not involve any change of sign and so does not allow any positive root. The system is therefore stable. This result is also true when both the fluids are viscous (Chandrasekhar [2]) or Oldroydian viscoelastic (Sharma [7]).

For the unstable configuration $\alpha_2 > \alpha_1$, there is at least one change of sign in equation (29) and so equation (29) has one positive root. The occurence of positive root implies that the system is unstable. The potentially unstable arrangement, therefore, remains unstable for the Rivlin-Ericksen elastico-viscous fluid.

We now examine the behaviour of growth rates with respect to kinematic viscosity and kinematic viscoelasticity analytically. Since $\alpha_2 > \alpha_1$ equation (29) has one positive root, let n_0 denote the positive root. Then

$$(1+2k^2\nu')n_0^2+(2k^2\nu)n_0-gk(\alpha_2-\alpha_1)=0. \qquad ...(30)$$

To study the behaviour of growth rates with respect to kinematic viscosity and kinematic viscoelasticity, we examine the natures of $\frac{dn_0}{d\nu}$ and $\frac{dn_0}{d\nu'}$ analytically.

If follows from equation (30) that

$$\frac{dn_0}{d\nu}=-\frac{k^2 n_0}{(1+2k^2\nu')n_0+\nu k^2}, \qquad ...(31)$$

and

$$\frac{dn_0}{d\nu'}=-\frac{k^2 n_0^2}{(1+2k^2\nu')n_0+\nu k^2}. \qquad ...(32)$$

From (31) and (32), we find that the growth rate decreases with the increase in kinematic viscosity as well as with the increase in kinematic viscoelasticity.

REFERENCES

[1] P.K. Bhatia, Rayleigh-Taylor instability of two viscous superposed conducting fluid, *Nuovo Cimento B(II)* **19** (1974), 161- 169.

[2] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Clarendon Press, Oxford, 1961.

[3] R.A. Gerwin, Hydromagnetic surface waves in a conducting liquid surrounded by a flowing gas, *Phys. Fluids* **11** (1968), 1699-1708.

[4] J.G. Oldroyd, Non-Newtonian effects in steady motion of some idealized elastico-viscous liquids, *Proc. Roy. Soc. London Ser. A* **245** (1958) 278-279.

[5] P.H. Roberts, The effect of a vertical magnetic field on Rayleigh-Taylor instability, *Astrophys. J.* **137** (1963), 679-689.

[6] R.C. Sharma and K.C. Sharma, Rayleigh-Taylor instability of two viscoelastic superposed fluids, *Acta Phys. Hungar,* **45** (1978), 213-220.

[7] R.C. Sharma, Instability of the plane interface between two viscoelastic superposed conducting fluids, *J.Math. Phys. Sci.* **12** (1978), 603-613.