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ON THE SUMMABILITY OF DOUBLE ORTHOGONAL SERIES

By

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Let $\phi_{ij}(x, y)$, $(i, j = 0, 1, 2, ...)$ be a sequence of orthonormal functions in the rectangle R $[a \le x \le b, c \le y \le d]$ i.e.

$$\int_R\int \phi_{ij}(x, y)\, \phi_{k,l}(x, y)\, dx\, dy = \begin{cases} 1 & \text{for } k = i, l = j \\ 0 & \text{for } k \ne i, l \ne j \end{cases}$$

Consider the double orthogonal series

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij}\, \phi_{ij}(x, y) \qquad \text{... (1.1)}$$

where $\{a_{ij}\}$ is an arbitrary sequence of real numbers.

The mnth partial sum of the series (1.1) is given by

$$s_{mn}(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij}\, \phi_{ij}(x, y).$$

The series (1.1) is said to be $(R, \lambda, \mu, 1, 1)$ summable to a function $s(x, y)$, if

$$\lim_{m, n \to} \sigma_{mn}(\lambda, \mu, x, y) = s(x, y)$$

where

$$\sigma_{mn}(\lambda, \mu, x, y) = \sum_{k=0}^{m} \sum_{l=0}^{n} \left(1 - \frac{\lambda_k}{\lambda_{m+1}}\right)\left(1 - \frac{\mu_l}{\mu_{n+1}}\right) a_{kl}\, \phi_{kl}(x, y)$$

The mnth Euler mean of the series (1.1) is given by

$$\tau_{mn}(x,y) = \frac{1}{2^{m+n}} \sum_{k=0}^{m} \sum_{l=0}^{n} \binom{m}{k}\binom{n}{l} s_{kl}(x,y).$$

The series (1.1) is said to be $(E,1,1)$ summable to a function $s(x,y)$, if

$$\lim_{m,n\to\infty} \tau_{mn}(x,y) = s(x,y).$$

The convergence and Casáro summability of single orthogonal series have been studied by Alexits[2], Kaczmarz[3], Menchoff[6], Rademacher[10], Tandori[12] and Zygmund[13] and those of double orthogonal series by Agnew[1], Fedulov[4], Pandzakidze[8], Mitchell[7], Patel[9] and Sapre[11]. Dealing with (C,1,1) summability of double orthogonal series (1.1) Patel[9] has proved the following theorem :

Theorem. If

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{kl}^2 < \infty \qquad \text{... (1.2)}$$

Then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn(\log m \log n)^p} (s_{mn}(x,y) - \sigma_{mn}(x,y))^2, p > 1$$

is convergent almost everywhere in R, where $\sigma_{mn}(x,y)$ is the $(C,1,1)$ mean of the series (1.1) We extend in this paper the above result to Riesz and Euler summability as follows :

Theorem 1. Under the condition (1.2)

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn(\log m \log n)^p} [s_{mn}(x,y) - \sigma_{mn}(\lambda, \mu, x, y)]^2, p > 1$$

converges almost everywhere in R.

Theorem 2. Under the condition (1.2)

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn(\log m \log n)^p} [s_{mn}(x,y) - \tau_{mn}(x,y)]^2, p > 1$$

converges almost everywhere in R.

In order to prove the above theorems, we need the following lemmas :

Lemma 1 [5]. Writing

$$W_{nk} = \frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i} - \frac{2k}{n+1},$$

we have

$$W_{nk} < 0 \left[\frac{n}{3} \right] + 2 \le k \le n.$$

Lemma 2 [5].

$$\frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i}^2 \le A \frac{k^2}{n^2} \text{ for } 1 \le k \le \left[\frac{n}{3} \right] + 1$$

2. Proof of Theorem 1. We have

$$s_{mn}(x, y) - \sigma_{mn}(\lambda, \mu, x, y)$$

$$= \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl}\, \phi_{k\lambda}(x, y) - \sum_{k=0}^{m} \sum_{l=0}^{n} \left(1 - \frac{\lambda_k}{\lambda_{m+1}}\right)\left(1 - \frac{\mu_1}{\mu_{n+1}}\right) a_{kl}\, \phi_{kl}(x, y)$$

$$= \sum_{k=0}^{m} \sum_{l=0}^{n} \left(\frac{\lambda_k}{\lambda_{m+1}} + \frac{\mu_l}{\mu_{n+1}} - \frac{\lambda_k \mu_l}{\lambda_{m+1} \mu_{n+1}} \right) a_{kl}\, \phi_{kl}(x, y)$$

$$= \sum_{k=1}^{m} \sum_{l=1}^{n} \frac{\lambda_k}{\lambda_{n+1}} a_{kl}\, \phi_{kl}(x, y) + \sum_{k=1}^{m} \sum_{l=1}^{n} \frac{\mu_1}{\mu_{n+1}} a_{kl}\, \phi_{kl}(x, y)$$

$$- \sum_{k=1}^{m} \sum_{l=1}^{n} \frac{\lambda_k \mu_i}{\lambda_{m+1} \mu_{n+1}} a_{kl}\, \phi_{kl}(x, y).$$

Now, from the inequality

$$(\alpha_1 + \alpha_2 + \alpha_3)^2 \le 3(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)$$

and the orthonormality property of the functions $\phi_{ij}(x, y)$, we get

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn (\log m \log n)^p} \int_R\!\!\int [s_{mn}(x, y) - \sigma_{mn}(\lambda, \mu, x, y)]^2\, dx\, dy$$

$$\leq 3 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn (\log m \log n)^p} \sum_{k=1}^{m} \sum_{l=1}^{n} \frac{\lambda_k^2}{\lambda_{m+1}^2} a_{kl}^2 +$$

$$+ 3 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn (\log m \log n)^p} \sum_{k=1}^{m} \sum_{l=1}^{n} \frac{\mu_l^2}{\mu_{n+1}^2} a_{kl}^2$$

$$+ 3 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn (\log m \log n)^p} \sum_{k=1}^{m} \sum_{l=1}^{n} \frac{\lambda_k^2 \mu_l^2}{\lambda_{m+1}^2 \mu_{n+1}^2} a_{kl}^2$$

$= 3\,\Sigma_1 + 3\,\Sigma_2 + 3\,\Sigma_3$, say.

Now,

$$\Sigma_1 = \sum_{m=1}^{\infty} \frac{1}{m (\log m)^p \lambda_{m+1}^2} \sum_{k=1}^{m} \lambda_k^2 \sum_{n=1}^{\infty} \frac{1}{n (\log n)^p} \sum_{l=1}^{n} a_{kl}^2$$

$$= \sum_{k=1}^{\infty} \lambda_k^2 \sum_{m=k}^{\infty} \frac{1}{m (\log m)^p \lambda_{m+1}^2} \sum_{l=1}^{\infty} a_{kl}^2 \sum_{n=l}^{\infty} \frac{1}{n (\log n)^p}$$

$$\leq \sum_{k=1}^{\infty} \sum_{m=k}^{\infty} \frac{1}{m (\log m)^p} \sum_{l=1}^{\infty} a_{kl}^2 \sum_{n=l}^{\infty} \frac{1}{n (\log n)^p}$$

$$= O(1) \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{kl}^2 < \infty .$$

The convergence of Σ_2 and Σ_3 follows on the same line.

Hence by B.Levy's theorem, it follows that the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn (\log m \log n)^p} [s_{mn}(x, y) - \sigma_{mn}(\lambda, \mu, x, y)]^2$$

converges almost everywhere in R.

This completes the proof of the theorem.

3. Proof of Theorem 2. We have

$s_{mn}(x, y) - \tau_{mn}(x, y)$

$$2^{m+n} \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl}\, \phi_{kl}(x, y) \sum_{l=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j}$$

$$-\frac{1}{2^{m+n}}\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\sum_{k=0}^{i}\sum_{l=0}^{j}a_{kl}\ \phi_{kl}(x,y)$$

$$=\frac{1}{2^{m+n}}\sum_{k=0}^{m}\sum_{l=0}^{n}a_{kl}\ \phi_{kl}(x,y)\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}$$

$$-\frac{1}{2^{m+n}}\sum_{k=0}^{m}\sum_{l=0}^{n}a_{kl}\ \phi_{kl}(x,y)\sum_{i=k}^{m}\sum_{j=l}^{n}\binom{m}{i}\binom{n}{j}$$

$$=\frac{1}{2^{m+n}}\sum_{k=0}^{m}\sum_{l=0}^{n}\left[\sum_{i=0}^{m}\binom{m}{i}\sum_{j=0}^{l-1}\binom{n}{j}+\sum_{i=0}^{k-1}\binom{m}{i}\sum_{j=l}^{n}\binom{n}{j}\right]a_{kl}\ \phi_{kl}(x,y)$$

$$=\frac{1}{2^{m+n}}\sum_{k=0}^{m}\sum_{l=0}^{n}2^{m}\sum_{j=0}^{l-1}\binom{n}{j}a_{kl}\ \phi_{kl}\ (x,y)$$

$$+\frac{1}{2^{m+n}}\sum_{k=0}^{m}\sum_{l=0}^{n}a_{kl}\ \phi_{kl}(x,y)\sum_{i=0}^{k-1}\binom{m}{i}\sum_{j=l}^{n}\binom{n}{j}.$$

Now, from the inequality

$$(\alpha_1+\alpha_2)^2\le 2\,(\alpha_1^2+\alpha_2^2)$$

and the orthonormality property of the functions $\{\phi_{ij} : (x, y)\}$, we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn\,(\log m\,\log n)^p}\int\!\!\int_R [s_{mn}(x,y)-\tau_{mn}(x,y)]^2\,dx\,dy$$

$$\le 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn\,(\log m\,\log n)^p}\sum_{k=0}^{m}\sum_{l=0}^{n}a_{kl}^2\left\{\frac{1}{2^n}\sum_{j=0}^{l-1}\binom{n}{j}\right\}^2$$

$$+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn\,(\log m\,\log n)^p}\sum_{k=0}^{m}\sum_{l=0}^{n}a_{kl}^2\left\{\frac{1}{2^{m+n}}\sum_{i=0}^{k-1}\binom{m}{i}\sum_{j=l}^{n}\binom{n}{j}\right\}^2$$

$$=O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn\,(\log m\,\log n)^p}\sum_{k=1}^{m}\sum_{l=1}^{n}a_{kl}^2\,\frac{l^2}{n^2}$$

$$+O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn\,(\log m\,\log n)^p}\sum_{k=1}^{m}\sum_{l=1}^{n}a_{kl}^2\,\frac{k^2}{m^2}$$

$$= O(1) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn\,(\log m \log n)^p} \sum_{k=1}^{m} \sum_{l=1}^{n} a_{kl}^2$$

$$= O(1) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m\,(\log m)^p} \sum_{l=1}^{\infty} a_{kl}^2 \sum_{n=l}^{\infty} \frac{1}{n\,(\log n)^p}$$

$$= O(1) \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{kl}^2 < \infty.$$

Hence, it follows by B.Levy's theorem that the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn\,(\log m \log n)^p} \left[s_{mn}(x,y) - \tau_{mn}(x,y)\right]^2$$

converges almost everywhere.

With this the theorem is proved.

REFERENCES

[1] R.P. Angew, On double orthogonal series, *Proc. London Math. Soc.* **33**(2) (1931-32), 420-434.

[2] G. Alexits, Ein summationssatz für Orthogonalreihen *Acta.. Math. Acad. Sci. Hungar.* **7** (1959) 5-9.

[3] S. Kaczmarz, Uber die summierbarkeit der Orthogonalreihen *Math. Zeit.* **26**, (1927), 99-105.

[4] V.S. Fedulov, On $(C,1,1)$ summability of double orthogonal series. *Ukrain Math. ZH* **7** (1955), 433-442.

[5] J. Meder, On the summability almost everywhere of orthogonal series by the method of Euler Knopp *Ann. Polon Math.* **5** (1958), 135-148.

[6] D. Menchoff, Sur les series de functions orthogonales I, *Fund. Math.* **4** (1923), 82-105.

[7] J. Mitchell, Convergence and $(C,1,1)$ - summability of double orthogonal series. *Duke. Math. Journal* **18** (1951), 211-219.

[8] S.P. Pandzakidze, On the theorem of Menchov and Rademacher for double orthogonal series. *Soobsc Akad. Nauk. Gruzin SSR.* **39** (1965), 277-282.

[9] C.M. Patel, On double orthogonal series *Proc. Nat. Acad. Sci.* Sec. A. **37** Part II (1967), 241-244.

[10] H. Rademacher, Einige sätze Über Reihen von allgemeinen orthogonal funktionen *Math. Annalen* **87** (1922) 112-138.

[11] A.R. Sapre, On Euler means and T-means of orthogonal series *Mat. Vesnik.* **9** (1972) 91-96.

[12] K. Tandori, über die orthogonalen Funktionen II (summation) *Acta. Sci. Math.* **18** (1957), 149-168.

[13] A. Zygmund, Sur l'application de la premiére movenne arithmetique dans la theoree séries de fonctions orthogonales *Fund. Math.* **10 (1927)**, 355-362.