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ON A SPECIAL KAEHLERIAN RECURRENT SPACE OF SECOND ORDER

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1. INTRODUCTION - An $n(=2m)$ dimensional Kaehlerian space K_n is a Riemannian space, which admits a structure field F_i^h, satisfying the relations (Yano 1965) :

$$F_i^h F_h^j = -\delta_i^j , \qquad \text{... (1.1)}$$

$$F_{ij} = -F_{ji}, \; (F_{ij} = F_i^a g_{aj}) \qquad \text{... (1.2)}$$

and

$$\nabla_j F_i^h = 0 , \qquad \text{... (1.3)}$$

where ∇ denotes covariant differentiation with respect to the linear symmetric connection of the space.

The Riemannian curvature tensor, which we denote by R_{ijk}^h is given by

$$R_{ijk}^h = \partial_i \left\{ {h \atop jk} \right\} - \partial_j \left\{ {h \atop ik} \right\} + \left\{ {h \atop ia} \right\} \left\{ {a \atop jk} \right\} - \left\{ {h \atop ja} \right\} \left\{ {a \atop ik} \right\}.$$

whereas the Ricci tensor and the scalar curvature are respectively given by $R_{ij} = R_{aij}^a$ and $R = R_{ij} g^{ij}$.

Let us, now, consider an affinely connected n-dimensional Kaehlerian recurrent space of second order, whose curvature tensor R_{ijk}^h satisfies the following condition :

$$\nabla_n \nabla_m R_{ijk}^h = \lambda_{mn} R_{ijk}^h , \qquad \text{... (1.4)}$$

where λ_{mn} is a non-symmetric, in general and non-vanishing covariant tensor. We shall assume to put the following two conditions in our space :

$$\nabla_j v^i = \phi_j v^i \qquad \text{... (1.5)}$$

and $$R_{jk} = \phi_j \alpha_k , \qquad \text{... (1.6)}$$

where α_k means a suitable covariant tensor, R_{kl}, we have

$$\nabla_n \nabla_m R_{ij} = \lambda_{mn} R_{ij} \qquad \text{... (1.7)}$$

Making use of (1.6) in (1.7), we have

$$\nabla_n \alpha_j \nabla_m \phi_i + \alpha_j \nabla_n \nabla_m \phi_i + \nabla_n \phi_i \nabla_m \alpha_j + \phi_i \nabla_n \nabla_m \alpha_j = \alpha_{mn} \phi_i \alpha_j \qquad \text{... (1.8)}$$

In fact, when the space under consideration admits an affine motion

$$\bar{x}^i = x^i + v^i \delta t.$$

Characterized by (1.5), we have a resolved form of Ricci-tensor of the form (1.6). Here, we assume the existence of recurrent cavariant vector v^i given by (1.5) and in addition, the resolvability of R_{ij}.

Making a commutator on the indices m and n in (1.8), we obtain

$$-\alpha_j \phi_a R^a_{imn} - \phi_i \alpha_a R^a_{jmn} = A_{mn} \phi_i \alpha_j \qquad \text{... (1.9)}$$

where $$A_{mn} \overset{\text{def}}{=\!=} \lambda_{mn} - \lambda_{nm}.$$

Multiplying (1.9) by v^i and summing over i and making use of (1.8), we have

$$\phi (A_{mn} \alpha_j + \alpha_a R^a_{jmn} + \alpha_j \Omega_{mn}) = 0,$$

where $$\phi \overset{\text{def}}{=\!=} \phi_i v^i.$$

In the present paper, we have to discuss the next two cases :

$$A_{mn} \alpha_j + \alpha_a R^a_{jmn} + \alpha_j \Omega_{mn} = 0 \qquad \text{... (1.10a)}$$

and $$\phi = 0. \qquad \text{... (1.10b)}$$

2. THE CASE OF $\alpha = 0$. Putting $\alpha = \alpha_m v^m$, the condition (1.10a) yields this case. We shall show this fact. Making use of (1.8) in (1.10a), we have

$$\alpha (A_{mn} + 2\Omega_{mn}) = 0.$$

Consequently, in the present case (1.10 a), we have to consider two cases. The first is a case of $\alpha = 0$ and the second is that of

$$A_{mn} + 2\Omega_{mn} = 0.$$

In the latter case, multiplying this condition by v^h and making use of (1.8), we have

$$A_{mn}\, v^h + 2R^h_{amn}\, v^a = 0,$$

from which, by contraction on h and n, it follows that

$$A_{mn}\, v^n + 2R_{am}\, v^a = 0,$$

or $$A_{mn}\, v^n + 2\phi\alpha_m = 0,$$

i.e., the second case means the first case and (1.10 a) may be replaced by $\alpha = 0$. By this reason, there exists only one case of $\alpha \neq 0$.

From (1.4), we have

$$A_{mn}\, R^h_{ijk} = R^h_{amn}\, R^a_{ijk} - R^a_{imn}\, R^h_{ajk} + R^a_{jmn}\, R^h_{ika} - R^a_{kmn}\, R^h_{ija} \quad \text{... (2.1)}$$

Contracting the indices h and k in (2.1), we have

$$A_{mn}\, R_{ij} = - R^a_{imn}\, R_{aj} - R^a_{jmn}\, R_{ia}\,. \quad \text{... (2.2)}$$

Multiplying (2.2) by v^i and summing on i, because of $v^i\, R_{ij} = v^i\, \phi_i\, \alpha_j = \phi\alpha_j$ and $\phi = 0$, we have

$$\alpha_a\, R^a_{jmn} = -\,(A_{mn} + \Omega_{mn})\alpha_j\,, \quad \text{... (2.3)}$$

where we have used (1.8) introducing (2.3) into the left hand side of (1.9), we have

$$\alpha_k\, (\phi_a\, R^a_{jmn} - \phi_j\, \Omega_{mn}) = 0.$$

Thus, there exist the following two cases to be discussed :

$$\alpha_k = 0 \quad \text{... (2.4a)}$$

$$\phi_a\, R^a_{jmn} = \phi_j\, \Omega_{mn} \quad \text{... (2.4b)}$$

In case of (2.4b), let us remember the first formula due to L.Bianchi, then we obtain

$$\phi_j\, \Omega_{mn} + \phi_m\, \Omega_{nj} + \phi_j\, \Omega_{jm} = 0. \quad \text{... (2.5)}$$

On the other hand, as we have

$$\Omega_{mn} = -\Omega_{nm} \text{ and } \phi = \phi_j v^j,$$

we derive $\Omega_{jk} v^i = -R_{ak} v^a = -\phi\alpha_k$.

from (1.8) by contraction on h and j.

Multiplying (2.5) by v^j and summing on j, we obtain

$$\Omega_{mn} = \alpha_m \phi_n - \alpha_n \phi_m . \quad \text{... (2.6)}$$

Remembering the definition of Ω_{mn} and (1.6), we can write down an interesting formula :

$$R_{jk} - R_{kj} = \nabla_j \phi_k - \nabla_k \phi_j .$$

As we have (2.4b) and (1.8), we see that

$$\phi R^a_{jmn} v^h = \phi_j \Omega_{mn} v^h \text{ and } R^h_{amn} v^a = \Omega_{mn} v^h .$$

Hence, we have

$$\phi_a R^a_{jmn} v^h = \phi_j R^h_{amn} v^a$$

and this means that

$$R^a_{jmn} \nabla_a v^h - R^h_{amn} \nabla_j v^a = 0,$$

i.e., we have actually

$$\nabla_n \nabla_m (\nabla_j v^i) - \nabla_m \nabla_n (\nabla_j v^i) = 0.$$

Consequently, we can imagine the existence of a gradient vector ρ_k and we are able to put

$$\nabla_k \nabla_j v^h = \rho_k \nabla_j v^h \quad \text{... (2.7)}$$

From (1.5) and (2.7), we can deduce the equation :

$$\nabla_k \phi_j + \phi_j \phi_k = \rho_k \phi_j .$$

Multiplying the above equation by v^j, we have the following :

$$\begin{aligned} \phi\rho_k &= v^j \nabla_k \phi_j + \phi\phi_k \\ &= \nabla_k (v^j \phi_j) - \phi_j \nabla_k v^j + \phi\phi_k \\ &= \nabla_k \phi - \phi_j \phi_k v^j + \phi\phi_k \\ &= \nabla_k \phi - \phi\phi_k + \phi\phi_k = \nabla_k \phi . \end{aligned}$$

In this way, the existence of ϕ_j is examined and we have here a characteristic condition on $\nabla_j v^h$:

$$\nabla_k \nabla_j v^h = \rho_k \nabla_j v^h, \rho_k = \frac{\nabla_k \phi}{\phi}.$$

On the other hand, in case of (2.4a), from (1.6), we obtain $R_{jk} = 0$.

Summerizing the above all conditions, we have the following :

Theorem (2.1) . In a n-dimensional Kaehlerian recurrence space of second order, admitting a contravariant vector v^h, characterized by

$$\nabla_j v^i = \phi_j v^i$$

and having a disjointed Ricci tensor of the form $R_{jk} = \phi_j \alpha_k$, there exist a case of $\alpha_k v^m = 0$.

In this case, if $\alpha_m = 0$, then we have the vanishing of Ricci tensor R_{jk} and if $\alpha_m \neq 0$, we have

$$R_{kl} - R_{lk} = \nabla_l \phi_k - \nabla_k \phi_l.$$

The mixed tensor $\nabla_j v^i$ itself is a recurrent one characterized by

$$\nabla_l \nabla_k v^i = \rho_l \nabla_k v^i$$

for a definite gradient vector $\rho_l = \nabla_l \phi/\phi$.

Remark . The latter part of the above theorem does not contain a case of ϕ_j being a gradient vector.

Such a case is a showing but worthless one, in fact, if ϕ_j will be a gradient vector, we have

$$\alpha_l \phi_k - \alpha_k \phi_l = \nabla_k \phi_l - \nabla_l \phi_k = 0,$$

from which we have

$$\lambda \phi_k v^k = \alpha_k v^k = \alpha = 0,$$

this is $\lambda\phi = 0$ from which being $\phi \neq 0$, it follows that $\lambda = 0$.

Hence, we have $\alpha_k = 0$.

This a contradiction, for we have $\nabla_l \phi_k + \phi_k \phi_l = 0$ at this moment and the case of ϕ_k being a gradient vector occurs.

3. The Case of $\phi = 0$. Let us consider the case of (1.10b). Then using the analogous method used in §2, multiplying (2.2) by v^j, we obtain the following formula :

$$\phi_a R^a_{jmn} = -(A_{mn} + \Omega_{mn})\,\phi_j \qquad \text{... (2.8)}$$

Substituting (2.8) into the left hand side of (1.9), we get

$$\phi_j\,(\alpha_a R^a_{kmn} - \alpha_k \Omega_{mn}) = 0.$$

Hence, we have here two cases to be discussed. They are

$$\phi_j = 0 \qquad \text{... (2.9a)}$$

and

$$\alpha_a R^a_{kmn} = \alpha_k \Omega_{mn} \qquad \text{... (2.9b)}$$

The case of (2.9a) yields one of $\nabla_i v^h = 0$ and $R_{ij} = 0$.

The case of (2.9b) may be treated as follows :

We have

$$\alpha_j \Omega_{mn} + \alpha_m \Omega_{nj} + \alpha_n \Omega_{jm} = 0,$$

from which by contraction of v^j, we have

$$\alpha\Omega_{mn} = 0,$$

because of $\Omega_{jk} v^j = -R_{ak} v^a = -\phi_a\alpha_k v^a = -\phi\alpha_k$.

From (1.8), by contraction on the indices h and j, and $\Omega_{mn} = -\Omega_{nm}$ and $\phi = \phi_j v^j$.

In this way, we obtain $\Omega_{mn} = 0$, say $\nabla_n \phi_m = \nabla_m \phi_n$.

Thus, we can state here the following :

Theorem 2.2 . When $\phi = 0$ in our space, there exist two cases. One of them is a case of satisfying $\nabla_j v^i = 0$ and $R_{jk} = 0$ and the other is a case of $\nabla_l \phi_k = \nabla_k \phi_i$.

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