Jñānābha ,Vol. 27, 1997

QUASI-HADAMARD PRODUCT OF CERTAIN ANALYTIC FUNCTIONS

By

Vinod Kumar

Department of Mathematics
Christ Church College, Kanpur-208001, India

ABSTRACT

We consider the class $F_p^c(\lambda, \alpha)$ of certain analytic functions. It is shown that if $f \in F_p^{c_1}(\lambda_1, \alpha_1)$, $0 \le \lambda_1 < 1$, and $g \in F_p^{c_2}(\lambda_2, \alpha_2)$, $\lambda_2 \ge 1$, then the quasi-Hadamard product $f*g$ belongs to the class $F_p^{c_1+c_2}(\lambda_1, \alpha_1)$ or $F_p^{c_1+c_2}(\lambda_1, \alpha_2)$ according as $\alpha_1 \le \alpha_2$ or $\alpha_1 \ge \alpha_2$, respectively. This result fills a gap in a recent result of Subhas S. Bhoosnurmath and S.R. Swamy [Pure Appl. Math. Sci. **35** (1992), 71- 77].

1. INTRODUCTION

Let T_p denote the class of functions of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \; p \in N, \; a_{p+n} \ge 0,$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$.

If $f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$ and $g(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n}$ be any two functions in T_p, then their quasi-Hadamard product, denoted by $f*g$, is defined by (see [3])

$$(f*g)(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n}$$

A function f in T_p belongs to the class $F_p^c(\lambda, \alpha)$ if and only if

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^c (p + n\lambda)\, a_{p+n} \le p - \alpha,$$

where $c \ge 0$, $\lambda \ge 0$ and $0 \le \alpha < p$.

It is easy to establish the following containment relations :

(*i*) $F_p^c(\lambda, \alpha_2) \subset F_p^c(\lambda, \alpha_1)$, if $\alpha_1 < \alpha_2$.

(*ii*) $F_p^c(\lambda_2, \alpha) \subset F_p^c(\lambda_1, \alpha)$, if $\lambda_1 < \lambda_2$.

(*iii*) $F_p^{c_2}(\lambda, \alpha) \subset F_p^{c_1}(\lambda, \alpha)$, if $c_1 < c_2$.

Bhoosnurmath and Swamy [1] introduced the above class and proved the following result :

Theorem. *If* $f \in F_p^{c_1}(\lambda_1, \alpha_1)$ *and* $g \in F_p^{c_2}(\lambda_2, \alpha_2)$, *then the quasi-Hadamard product* $f*g$ *belongs to* $F_p^{c_0}(\lambda_0, \alpha_0)$, *where*

$F_p^{c_0}(\lambda_0, \alpha_0) \equiv F_p^{c_1+c_2+1}(\lambda_1, \alpha_1) \cap F_p^{c_1+c_2+1}(\lambda_2, \alpha_2)$, *if* $\lambda_1, \lambda_2 \geq 1$
and
$F_p^{c_0}(\lambda_0, \alpha_0) \equiv F_p^{c_1+C_2}(\lambda_1, \alpha_1) \cap F_p^{c_1+c_2}(\lambda_2, \alpha_2)$, *if* $0 \leq \lambda_1, \lambda_2 < 1$.

The technique used by Bhoosnurmath and Swamy [1] is the same as employed by the author [2].

We note that their is a gap in the above result, *i.e.*, the case when any one of λ_1 and λ_2 is less than 1 and the other is greater than or equal to 1. It is the purpose of this paper to fill up this gap.

2. MAIN RESULT

Theorem. *If* $f \in F_p^{c_1}(\lambda_1, \alpha_1)$ *and* $g \in F_p^{c_2}(\lambda_2, \alpha_2)$, *where* $0 \leq \lambda_1 < 1$ *and* $\lambda_2 \geq 1$, *then the quasi-Hadamard product* $f*g$ *belongs to* $F_p^{c_1+c_2+1}(\lambda_1, \alpha_1) \cap F_p^{c_1+c_2}(\lambda_2, \alpha_2)$.

Proof. $f \in F_p^{c_1}(\lambda_1, \alpha_1)$ implies $f \in F_p^{c_1}(0, 0)$ so that

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^{c_1} a_{p+n} \leq 1.$$

Therefore

$$\left(\frac{n+p}{p}\right)^{c_1} a_{p+n} \leq 1, \quad \text{for all } n = 1, 2, 3, \ldots \qquad \ldots (2.1)$$

Further, $g \in F_p^{c_2}(\lambda_2, \alpha_2)$ implies that $g \in F_p^{c_2}(1, 0)$ so that

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^{c_2+1} b_{p+n} \leq 1.$$

Therefore

$$\left(\frac{n+p}{p}\right)^{c_2+1} b_{p+n} \le 1, \text{ for all } n = 1, 2, 3, \dots \qquad \dots (2.2)$$

Using (2.2) we get

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^{c_1+c_2+1} (p+n\lambda_1)\, a_{p+n}\, b_{p+n}$$

$$\le \sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^{c_1} (p+n\lambda_1)\, a_{p+n}$$

$$\le p - \alpha_1, \quad \text{since } f \in F_p^{c_1}(\lambda_1, \alpha_1).$$

This shows that $f*g \in F_p^{c_1+c_2+1}(\lambda_1, \alpha_1)$.

Similarly, by using (2.1) we get

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^{c_1+c_2} (p+n\lambda_2)\, a_{p+n}\, b_{p+n}$$

$$\le \sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^{c_2} (p+n\lambda_2)\, b_{p+n}$$

$$\le p - \alpha_2, \quad \text{since } g \in F_p^{c_2}(\lambda_1, \lambda_2).$$

This shows that $f*g \in F_p^{c_1+c_2}(\lambda_2, \alpha_2)$.

Hence $f*g \in F_p^{c_1+c_2+1}(\lambda_1, \alpha_1) \cap F_p^{c_1+c_2}(\lambda_2, \alpha_2)$.

Corollary. *If* $f \in F_p^{c_1}(\lambda_1, \alpha_1)$ *and* $g \in F_p^{c_2}(\lambda_2, \alpha_2)$, *where* $0 \le \lambda_1 < 1$ *and* $\lambda_2 \ge 1$, *then the quasi-Hadamard product* $f*g$ *belongs to* $F_p^{c_1+c_2}(\lambda_1, \alpha_1)$ *or* $F_p^{c_1+c_2}(\lambda_1, \alpha_2)$ *according as* $\alpha_1 \le \alpha_2$ *or* $\alpha_1 \ge \alpha_2$.

Proof. We have

$$F_p^{c_1+c_2+1}(\lambda_1, \alpha_1) \cap F_p^{c_1+c_2}(\lambda_2, \alpha_2)$$

$$\subset F_p^{c_1+c_2}(\lambda_1, \alpha_1) \cap F_p^{c_1+c_2}(\lambda_2, \alpha_2)$$

$$\subset F_p^{c_1+c_2}(\lambda_1, \alpha_1) \cap F_p^{c_1+c_2}(\lambda_1, \alpha_2)$$

$$= F_p^{c_1+c_2}(\lambda_1, \alpha_1) \text{ or } F_p^{c_1+c_2}(\lambda_1, \alpha_2)$$

according as $\alpha_1 \le \alpha_2$ or $\alpha_1 \ge \alpha_2$.

The result follows now from the above theorem.

REFERENCES

[1] Subhas S. Bhoosnurmath and S.R. Swamy. *Modified Hadamard product of certain analytic functions*, Pure Appl. Math. Sci. **35**(1992), 71-77.

[2] Vinod Kumar. *Hadamard product of certain starlike fucntions*, J. Math. Anal. Appl. **110**(1985), 425-428.

[3] Vinod Kumar. *Quasi-Hadamard product of certain univalent functions*, J. Math. Anal. Appl. **126**(1987), 70-77.