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SOME PROBABILITY DISTRIBUTIONS AND EXPECTATIONS ASSOCIATED WITH MULTIAVRIATE BETA AND GAMMA DISTRIBUTIONS INVOLVING MULTIPLE HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES

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ABSTRACT

In the present paper, we discuss some multivariate beta and gamma distributions and make their applications to derive their expectations in terms of multiple hypergeometric functions of several variables.

1. Introduction. Exton [6, p.222] studied many special multivariate distributions having expectations in terms of Lauricella's multiple hypergeometric functions [8]. Motivated by this work, here in the present paper, we discuss some multivariate beta and gamma distributions and make their applications to derive the expectations of different functions in terms of multiple hypergeometric functions ${}^{(k)}_{(1)}E^{(n)}_{C}$ of Chandel [1], ${}^{(k)}_{(1)}E^{(n)}_{D}$, ${}^{(k)}_{(2)}E^{(n)}_{D}$ of Exton [5,6], ${}^{(k)}F^{(n)}_{AD}$, ${}^{(k)}F^{(n)}_{BD}$ of Chandel and Gupta [3] and ${}^{(k)}F^{(n)}_{CD}$ of Karlsson [7]. Finally we also discuss their special cases.

2. Expectations of different functions related to multivariate beta distributions. The expectation for the function $f(x_1, \dots, x_n)$ having multivariate density function $f(x_1, \dots, x_n)$ is defined as

$$(2.1) \quad < g\,(x_1, \dots, x_n) > = \int_{-\infty}^{\infty} \dots (n) \dots \int_{-\infty}^{\infty} f(x_1, \dots, x_n) g\,(x_1, \dots, x_n)\, dx_1 \dots dx_n$$

We consider the density function

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$$(2.2)\ f_1(x_1, \dots, x_n) = x_1^{b_1-1} \dots x_n^{b_n-1}(1-x_1-\dots-x_n)^{c-b_1-\dots-b_k-1}$$

$$\cdot (1-x_{k+1}-\dots-x_n)^{c'-b_{k+1}-\dots-b_n-1}$$

$$\cdot \frac{\Gamma(c)\Gamma(c')}{\Gamma(b_1)\dots\Gamma(b_n)\Gamma(c-b_1-\dots-b_k)\Gamma(c'-b_{k+1}-\dots-b_n)},$$

provided that $0 \le x_1, \dots, 0 \le x_n$, $x_1 + \dots + x_k \le 1$, $x_{k+1} + \dots + x_n \le 1$ and real parts of c, c', $c-b_1-\dots-b_k$ and $c'-b_{k+1}-\dots-b_n$ are positive and $f_1 = 0$ elsewhere.

Another density function is

$$(2.3)\ f_2(x_1, \dots, x_n) = x_1^{b_1-1} \dots x_n^{b_n-1}(1-x_1-\dots-x_k)^{c-b_1-\dots-b_k-1}$$

$$\cdot (1-x_{k+1})^{c_{k+1}-b_{k+1}-1} \dots (1-x_n)^{c_n-b_n-1}$$

$$\cdot \frac{\Gamma(c)\,\Gamma(a_{k+1}) \dots \Gamma(a_n)}{\Gamma(b_1)\dots\Gamma(b_n)\Gamma(c-b_1-\dots-b_k)\Gamma(c_{k+1}-b_{k+1}) \dots \Gamma(c_n-b_n)}$$

where $x_1 + \dots + x_k \le 1$, $0 \le x_r \le 1$, $r=1,\dots,n$

$\mathrm{Re}(c) > \mathrm{Re}(b_1 + \dots + b_k)$, $\mathrm{Re}(b_j) > 0$, j = 1, ..., k,

$\mathrm{Re}(c_i) > \mathrm{Re}(b_i) > 0$, $i=k+1,\dots,n$ and f_2 is zero elsewhere.

Corresponding to density function f_1 defined by (2.2) consider the function

$$(2.4)\qquad g_1(x_1, \dots, x_n) = (1-x_1\alpha_1 - \dots - x_n\alpha_n)^{-a}.$$

Now putting the values of f_1 and g_1 from (2.2) and (2.4) respectively in (2.1) and making an appeal to the result due to Exton [6, p.93. (3.4.2.4)] we obtain the expectation of g_1 corresponding to function f_1 :

$$(2.5)\quad < E_{1,1} > {}^{(k)}_{(1)}E_D^{(n)}(a, b_1, \dots, b_n; c, c'; \alpha_1, \dots, \alpha_n),$$

where ${}^{(k)}_{(1)}E_D^{(n)}$ is multiple hypergeometric function related to Lauricella's $F_D^{(n)}$ introduced by Exton [5].

Further putting the values of f_2 and g_1 from (2.3) and (2.4) respectively in (2.1) and making an appeal to the result due to Karlsson [7.(3.2)], we derive following expectation value of g_1 corresponding to the function f_2

(2.6) $< E_{1,2} > {}^{(k)}F_{AD}^{(n)} (a, b_1, \ldots, b_n ; c, c_{k+1}, \ldots, c_n ; \alpha_1, \ldots, \alpha_n)$

where ${}^{(k)}F_{AD}^{(n)}$ is one of intermediate Lauricella's multiple hypergeometric function introduced by Chandel and Gupta [3].

Now we consider the density function

$$(2.7)\ f_3 (x_1, \ldots, x_n) = x_1^{b_1-1} \ldots x_n^{b_n-1} (1-x_1-\ldots-x_n)^{c-b_1-\ldots-b_n-1} \cdot \frac{\Gamma(c)}{\Gamma(b_1)\ldots\Gamma(b_n)\Gamma(c-b_1-\ldots-b_n)}$$

provided that $0 \le x_1, \ldots, 0 \le x_n, x_1 + \ldots + x_n \le 1$ and all the real parts of c and $c - b_1 - \ldots - b_n$ are positive, also $f_3 = 0$ else where.

Consider

$$(2.8)\ g_2 (x_1, \ldots, x_n) = (1-\alpha_1 x_1 - \ldots - \alpha_k x_k)^{-a} (1 - \alpha_{k+1} x_{k+1} - \ldots - \alpha_n x_n)^{-a'}.$$

Then putting these values of f_3 and g_2 from (2.7) and (2.8) respectively in (2.1) and making an appeal to the result due to Exton [6, p.93. (3.4.2.5)], we derive the following expectation of g_2 corresponding to function f_3 :

$$(2.9)\quad < E_{2,3} > = {}^{(k)}_{(2)}E_D^{(n)} (a, a', b_1, \ldots, b_n ; c; \alpha_1, \ldots, \alpha_n)$$

where ${}^{(k)}_{(2)}E_D^{(n)}$ is another multiple hypergeometric function related to Lauricella's $F_D^{(n)}$ introduced by Exton [5].

Further consider the function

$$(2.10)\ g_3 (x_1, \ldots, x_n) = (1-\alpha_1 x_1 - \ldots - \alpha_k x_k)^{-a} (1-\alpha_{k+1} x_{k+1})^{-a_{k+1}} \ldots (1 - \alpha_n x_n)^{-a_n}.$$

Now putting the value of f_3 and g_3 from (2.7) and (2.10) respectively in (2.1) and making an appeal to the result due to Karlsson [7,(3.1)], we get the expectation value of g_3 corresponding to the function f_3

$$(2.11) < E_{3,3} > = {}^{(k)}F_{BD}^{(n)} [a, a_{k+1}, \ldots, a_n, b_1, \ldots, b_n ; c ; \alpha_1, \ldots, \alpha_n],$$

where ${}^{(k)}F_{BD}^{(n)}$ is one of intermediate Lauricella's multiple hypergeometric functions due to Chandel and Gupta [3].

3. Expectations of different multiple hypergeometric functions related to multivariate gamma distributions. In this section, we discuss some density functions associated with gamma distributions and derive some expectations involving multiple hypergeometric functions.

Consider the density function

$$(3.1)\quad f(x,y,z) = \frac{1}{\Gamma(a)\Gamma(a')\Gamma(b)} e^{-x-y-z} x^{a-1} y^{a'-1} z^{b-1},$$

provided that $0 \le x,y,z < \infty$, $\mathrm{Re}(a) > 0$, $\mathrm{Re}(a')>0$, $\mathrm{Re}(b)>0$ and $f(x,y,z)=0$ elsewhere.

Consider another function

$$(3.2)\quad g(x,y,z) = {}_0F_1(-; c_1; x_1 xz) \ldots {}_0F_1(-; c_k; x_k xz)$$
$${}_0F_1(-; c_{k+1}; x_{k+1} yz) \ldots {}_0F_1(-; c_n; x_n yz).$$

Now making an appeal to the result due to Exton [6, p.96 (3.4.4.6)], we derive the following expectations value of g(x, y, z) corresponding to the function f(x, y, z) :

$$(3.3)\quad < g(x, y, z) > = {}^{(k)}_{(1)}E_C^{(n)} (a,a', b; c_1, \ldots, c_n; x_1, \ldots, x_n),$$

where ${}^{(k)}_{(1)}E_C^{(n)}$ is multiple hypergeometric function related to Lauricella's $F_C^{(n)}$ introduced by Chandel [1].

Also consider the density function

$$(3.4)\quad f_4(y, x_1, \ldots, x_n) = \frac{1}{\Gamma(a)\Gamma(b_1)\ldots\Gamma(b_n)} e^{-y-x_1-\ldots-x_n} y^{a-1} x_1^{b_1-1} \ldots x_n^{b_n-1},$$

provided that $0 < y, x_1, \ldots, x_n < \infty$, $\mathrm{Re}(a)$, $\mathrm{Re}(b_1), \ldots, \mathrm{Re}(b_n) > 0$ and $f_4 = 0$ elsewhere.

Further consider

$$(3.5)\quad f_4(y, x_1, \ldots, x_n) = {}_0F_1[-, c; \alpha_1 xy + \ldots + \alpha_k x_k y]$$
$${}_0F_1[-, c'; \alpha_{k+1} x_{k+1} y + \ldots + \alpha_n x_n y].$$

Now putting the values of f_4 and g_4 from (3.4) and (3.5) respectively in (2.1) and making an appeal to the result due to Chandel and Gupta [2, (2.2)], we obtain the following expectation of g_4 corresponding to the function f_4 :

$$(3.6)\quad < E_{4,4} > {}^{(k)}_{(1)}E_D^{(n)} (a, b_1, \ldots, b_n; c, c'; \alpha_1, \ldots, \alpha_n).$$

Now we consider the density function

(3.7) $f_5(z,y,x_1, \ldots, x_n) = \dfrac{1}{\Gamma(a)\Gamma(a')\Gamma(b_1)\ldots\Gamma(b_n)} e^{-z-y-x_1- \ldots -x_n}$

$$z^{a-1} y^{a'-1} x_1^{b_1-1} \ldots x_n^{b_n-1},$$

where $0 < z,y,x_1, \ldots, x_n < \infty$ and all Re (a) Re (a'), Re(b_1),...,Re$(b_n) > 0$ also $f_5 = 0$ elsewhere.

Consider the density function

(3.8) $g_5(z,y,x_1, \ldots, x_n) = {}_0F_1[-, c ; (\alpha_1 x_1 + \ldots + \alpha_k x_k) z$

$$+ (\alpha_{k+1} x_{k+1} + \ldots + \alpha_n x_n) y].$$

Now putting the values of f_5 and g_5 from (3.7) and (3.8) respectively in (2.1) and then making an appeal to the result due to Chandel and Gupta [2, (2.3)] we get the following expectation of g_5 corresponding to the function f_5 :

(3.9) $< E_{5,5} > {}^{(k)}_{(2)}E_D^{(n)} (a,a', b_1, \ldots, b_n ; c, ; \alpha_1, \ldots, \alpha_n).$

Further consider the density function

(3.10) $f_6(x,x_{k+1}, \ldots, x_n, y_1, \ldots, y_n) = \dfrac{1}{\Gamma(a)\Gamma(a_{k+1})\ldots\Gamma(a_n)\Gamma(b_1)\ldots\Gamma(b_n)}$

$$x^{a-1} x_{k+1}^{a_{k+1}-1} \ldots x_n^{a_n-1} y_1^{b_1-1} \ldots y_n^{b_n-1} e^{-(x+x_{k+1}+\ldots+x_n+y_1+\ldots+y_n)},$$

where $0 \le x,x_{k+1},\ldots,x_n, y_1,\ldots,y_n < \infty$ and Re(a), Re (a_{k+1}), ..., Re(a_n), Re (b_1), ..., Re$(b_n) > 0$, also $f_6 = 0$ elsewhere.

Consider another density function

(3.11) $g_6(x,x_{k+1}, \ldots, x_n, y_1, \ldots, y_n)$

$$= {}_0F_1[-, c ; \alpha_1 x_1 x + \ldots + \alpha_k x_k x + \alpha_{k+1} x_{k+1} y_{k+1} + \ldots + \alpha_n x_n y_n].$$

Now putting the values of f_6 and g_6 from (3.10) and (3.11) respectively in (2.1) and making an appeal to the result due to Chandel and Gupta [3,(5.5)], we obtain the following expectation of g_6 having density function f_6 :

(3.12) $< E_{6,6} > {}^{(k)}F_{BD}^{(n)} [a,a_{k+1}, \ldots, a_n, b_1, \ldots, b_n ; c ; \alpha_1, \ldots, \alpha_n].$

Further consider density function

(3.13) $f_7(z,y,x_1, \ldots, x_k) = \frac{1}{\Gamma(a)\Gamma(b)\Gamma(b_1)\ldots\Gamma(b_k)} e^{-z-y-x_1-\ldots-x_k}$

$$. z^{a-1} y^{b-1} x_1^{b_1-1} \ldots x_k^{b_k-1},$$

provided that $0 \le z,y,x_1, \ldots, x_k < \infty$ and $\mathrm{Re}(a)$, $\mathrm{Re}(b)$, $\mathrm{Re}(b_1)$, ..., $\mathrm{Re}(b_k) > 0$, also $f_7 = 0$ elsewhere and

(3.14) $g_7(z,y,x_1, \ldots, x_k) = {}_0F_1(-, c ; \alpha_1 x_1 z + \ldots + \alpha_k x_k z)$

$$ {}_0F_1(- ; c_{k+1} ; zy\alpha_{k+1}) \ldots {}_0F_1(- ; c_n ; zy\alpha_n).$$

Now putting the values of f_7 and g_7 from (3.13) and (3.14) respectively in (2.1) and applying the result due to Chandel and Vishwakarma [4, p.178(3.9)], we derive corresponding expectation of g_7 :

(3.15) $< E_{7,7} > {}^{(k)}F_{CD}^{(n)}(a,b, b_1, \ldots, b_k ; c, c_{k+1}, \ldots, c_n ; \alpha_1, \ldots, \alpha_n)$

where ${}^{(k)}F_{CD}^{(n)}$ is fourth possible intermediate Lauricella's multiple hypergeometric function due to Karlsson [8].

4. Special Cases.

(4.1) For $k = 0$, from (2.5) and (2.9), we deduce separately the result due to Exton [6, (7.2.1.?)].

(4.2) For $k = 0$, from (2.6), we derive the result due to Exton [6, (7.2.3.1)],

while

(4.3) for $k = n$, from (2.?), we obtain the result due to Exton [6,(7.2.1.5)]

(4.4) for $k = 0$, (2.11) gives the result due to Exton [6, (7.2.1.6)],

while

(4.5) for $k = n$, (2.11) gives the result due to Exton [6, (7.2.1.5)].

Further

(4.6) for $k = 0$, from (3.6) and (3.9), we deduce the result due to Exton [6, (7.2.3.2)].

Also

(4.7) for $k = n$, from (3.12) and (3.15), we derive the result due to Exton [6, (7.2.3.2)].

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