Jñānābha, Vol. 27, 1997

SOME DOUBLE INTEGRALS AND LAPLACE TRANSFORMS ASSOCIATED WITH THEM

By

Mridula Garg and Mahesh Kumar Gupta
Department of Mathematics, University of Rajasthan,
Jaipur - 302 004, India

(Received : December 5, 1996)

ABSTRACT

In the present paper, we derive two infinite double integrals involving the exponential function, the product of general polynomials $S_V^U[x]$, the polynomial set $S_n^{\alpha,\beta,0}[x]$ and the H-function of two variables. Due to the presence of generalized polynomials and functions, the results obtained here are quite general in nature and are capable of yielding a large number of known and new integrals and double Laplace transform of certain functions. To illustrate we have obtained here two particular cases of our result. One of them gives the double Laplace transform of product of two generalizations of Hermite polynomials given by Gould and Hopper [4] and the other is an integral involving the product of Laguerre polynomials $L_n^{(\alpha)}(x)$ and a generalization of Laguerre polynomials $L_n^{(\alpha)}(x, r, \beta)$ introduced by Singh and Srivastava [11].

1. INTRODUCTION

(i) The two dimensional Laplace transform of a function $f(x_1, x_2)$ will be defined as

$$L\{f(x_1, x_2) : s_1, s_2\} = \int_0^\infty \int_0^\infty e^{-s_1x_1 - s_2x_2} f(x_1, x_2)\, dx_1\, dx_2 \qquad \text{... (1.1)}$$

(ii) The general class of polynomials $S_V^U[x]$ introduced by Srivastava [12, p. 1, eq. (1)]

$$S_V^U[x] = \sum_{K=0}^{[V/U]} \frac{(-V)_{UK} A_{V,K}\, x^K}{K!} \quad (K = 0, 1, 2, \ldots) \qquad \text{... (1.2)}$$

where U is an arbitrary positive integer and the coefficients $A_{V,K}$ $(V, K \geq 0)$ are arbitrary constants, real or complex. By suitably specializing the coefficients $A_{V,K}$, the polynomials $S_V^U[x]$ can easily be reduced to the classical orthogonal polynomials including for example, the Laguerre polynomials $L_n^{(\alpha)}(x)$, the generalized Hermite polynomials $g_V^U(x, h)$ and several others (see, for details [14, p.158-161]).

(iii) The generalized polynomial set $S_n^{\alpha, \beta, \tau}[x]$ is defined by following Rodrigues type formula [9,p.64,eq.(2.1.8)]

$$S_n^{\alpha, \beta, \tau}[x : r, s, q, A, B, m, k, l]$$
$$= (Ax+B)^{-\alpha}(1-\tau x^r)^{-\beta/\tau}\, T_{k,l}^{m+n}\left[(Ax+B)^{\alpha+qn}(1-\tau x^r)^{\beta/\tau+sn}\right] \quad \dots (1.3)$$

with the differential operator $T_{K,l}$ being defined as

$$T_{k,l} \equiv x^l\left(k + x\frac{d}{dx}\right) \quad \dots (1.4)$$

The explicit form of this generalization polynomial set [9,p.71, eq.(2.3.4)] is given by

$$S_n^{\alpha, \beta, \tau}[x : r, s, q, A, B, m, k, l] = B^{qn}\, x^{l(m+n)}\,(1-\tau x^r)^{sn}\, l^{m+n}$$

$$\sum_{v=0}^{m+n}\sum_{u=0}^{v}\sum_{j=0}^{m+n}\sum_{i=0}^{j} \frac{(-1)^j(-j)_i(\alpha)_j(-v)_u(-\alpha-qn)_i}{u!\,v!\,i!\,j!\,(1-\alpha-j)_i}$$

$$\left(-\frac{\beta}{\tau}-sn\right)_v\left(\frac{i+k+ru}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^v\left(\frac{Ax}{B}\right)^i \quad \dots (1.5)$$

It may be pointed out here that the polynomial set defined by (1.2) is very general in nature and it unifies and extends a number of classical polynomials introduced and studied by various research workers such as Chatterjea [1], Dhillon [3], Gould- Hopper [4], Krall and Frink [7], Singh [10] and Singh and Srivastava [11] etc. Some of the special cases of (1.4) are given by Raizada in a tabular form [9,p.65]. We shall require the following special case of (1.4).

If we take $A = 1$, $B = 0$ and let $\tau \to 0$ in (1.4) and use the well known confluence principle

$$\lim_{|b|\to\infty} (b)_n\,(x/b)^n = x^n \quad \dots (1.6)$$

therein, we arrive at the following polynomial set

$$S_n^{\alpha,\beta,0}[x] = S_n^{\alpha,\beta,0}\,[x : r, q, l, 0, m, k, l]$$

$$= x^{an+l(m+n)} l^{m+n} \sum_{v=0}^{m+n} \sum_{u=0}^{v} \frac{(-v)_u \left(\dfrac{\alpha+an+k+ru}{l}\right)_{m+n} (\beta x^r)^v}{u!\,v!} \quad \dots (1.7)$$

(iv) The H-function of two variables defined by Mittal and Gupta [8,p.117, eq.(1.1.1)], possesses the following integral representation

$$H[x,y] = H^{O,N:M_1,N_1:M_2,N_2}_{P,Q:P_1,Q_1:P_2,Q_2}\left[\begin{matrix} x \\ y \end{matrix}\middle|\begin{matrix} (a_j:\alpha_j',\alpha_j'')_{1,p} : \\ (b_j:\beta_j',\beta_j'')_{1,Q} : \end{matrix}\ \begin{matrix} (c_j',\gamma_j')_{1,P_1} : (c_j'',\gamma_j'')_{1,P_2} \\ (d_j',\delta_j')_{1,Q_1} : (d_j'',\delta_j'')_{1,Q_2} \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2} \phi(\xi,\eta)\ \phi_1(\xi)\,\phi_2(\eta)\,x^{\xi}y^{\eta}\,d\xi\,d\eta,\ \omega=\sqrt{-1} \dots (1.8)$$

where

$$\phi(\xi,\eta) = \frac{\prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j'\xi+\alpha_j''\eta)}{\prod_{j=N+1}^{p}\Gamma(a_j-\alpha_j'\xi+\alpha_j''\eta)\ \prod_{j=1}^{Q}\Gamma(1-b_j+\beta_j'\xi+\beta_j''\eta)} \dots (1.9)$$

$$\phi_1(\xi) = \frac{\prod_{j=1}^{M_1}\Gamma(d_j'-\delta_j'\xi)\ \prod_{j=1}^{N_1}\Gamma(1-c_j'+\gamma_j'\xi)}{\prod_{j=M_1+1}^{Q_1}\Gamma(1-d_j'+\delta_j'\xi)\ \prod_{j=N_1+1}^{P_1}\Gamma(c_j'-\gamma_j'\xi)} \dots (1.10)$$

and
$$\phi_2(\eta) = \frac{\prod_{j=1}^{M_2}\Gamma(d_j''-\delta_j'')\eta\ \prod_{j=1}^{N_2}\Gamma(1-c_j''+\gamma_j''\eta)}{\prod_{j=M_2+1}^{Q_2}\Gamma(1-d_j''+\delta_j''\eta)\ \prod_{j=N_2+1}^{P_2}\Gamma(c_j''-\gamma_j''\eta)} \dots (1.11)$$

For the convergence, existence conditions and other details of the H-function of two variables, we refer to the book [13,p.83, eq.(6.1.5), (6.1.6)].

2. FIRST INTEGRAL

$$\int_0^\infty\int_0^\infty (\lambda_1x_1+\lambda_2x_2)^{\sigma_1-1}(\mu_1x_1+\mu_2x_2)^{\sigma_2-1}\exp\,[-s_1(\lambda_1x_1+\lambda_2x_2)-s_2(\mu_1x_1+\mu_2x_2)]$$
$$S_V^U\,[\lambda_1x_1+\lambda_2x_2]\quad S_n^{\alpha,\beta,0}\quad[\mu_1x_1+\mu_2x_2]\ dx_1\,dx_2$$

$$= \frac{1}{R} \sum_{K=0}^{[V/U]} \sum_{v=0}^{m+n} \sum_{u=0}^{v} \theta(K, u, v) \frac{\Gamma(\sigma_1 + K)\, \Gamma(\sigma_2 + qn + l(m+n) + vr)}{s_1^{\sigma_1 + k}\, s_2^{\sigma_2 + qn + (m+n) + vr}} \quad \ldots (2.1)$$

where

$$\theta(K, u, v) = \frac{(-V)_{UK} A_{V,K}\, l^{m+n} (-v)_u\, \beta^v}{K!\, u!\, v!} \left(\frac{\alpha + qn + k + ru}{l}\right)_{m+n} \quad \ldots (2.2)$$

and

$$R = \begin{vmatrix} \lambda_1, \mu_1 \\ \lambda_2, \mu_2 \end{vmatrix} \neq 0, \mathrm{Re}(\sigma_1) > 0, \mathrm{Re}(s_i) > 0 \ (i = 1, 2,) \quad \ldots (2.3)$$

Proof : To prove (2.1), we make use of the known double integral [15,p.241, eq.(7)]

$$\int_0^\infty \int_0^\infty F(\lambda_1 x_1 + \lambda_2 x_2, \mu_1 x_1 + \mu_2 x_2)\, dx_1\, dx_2 = \frac{1}{R} \int_0^\infty \int_0^\infty F(u_1, u_2)\, du_1\, du_2 \quad \ldots (2.4)$$

where R is defined as in (2.3).

If we take

$$F(\lambda_1 x_1 + \lambda_2 x_2, \mu_1 x_1 + \mu_2 x_2) = f_1(\lambda_1 x_1 + \lambda_2 x_2)\, f_2(\mu_1 x_1 + \mu_2 x_2)$$

then $\int_0^\infty \int_0^\infty f_1(\lambda_1 x_1 + \lambda_2 x_2)\, f_2(\mu_1 x_1 + \mu_2 x_2)\, dx_1\, dx_2$

$$= \frac{1}{R} \int_0^\infty f_1(u_1)\, dx_1 \int_0^\infty f_2(u_2) du_2 \quad \ldots (2.5)$$

Consider

$$f_1(\lambda_1 x_1 + \lambda_2 x_2) = (\lambda_1 x_1 + \lambda_2 x_2)^{\sigma_1 - 1} \exp[-s_1(\lambda_1 x_1 + \lambda_2 x_2)]\, S_V^U [\lambda_1 x_1 + \lambda_2 x_2]$$

and $f_2(\mu_1 x_1 + \mu_2 x_2) = (\mu_1 x_1 + \mu_2 x_2)^{\sigma_2 - 1} \exp[-s_2(\mu_1 x_1 + \mu_2 x_2)]\, S_n^{\alpha, \beta, 0} [\mu_1 x_1 + \mu_2 x_2]$

Then, from (2.5), we have

$$\int_0^\infty \int_0^\infty (\lambda_1 x_1 + \lambda_2 x_2)^{\sigma_1 - 1} (\mu_1 x_1 + \mu_2 x_2)^{\sigma_2 - 1} \exp[-s_1(\lambda_1 x_1 + \lambda_2 x_2) - s_2(\mu_1 x_1 + \mu_2 x_2)]\, S_V^U [\lambda_1 x_1 + \lambda_2 x_2]\, S_n^{\alpha, \beta, 0} [\mu_1 x_1 + \mu_2 x_2]\, dx_1\, dx_2$$

$$= \frac{1}{R} \int_0^\infty u_1^{\sigma_1 - 1} e^{-s_1 u_1} S_V^U [u_1]\, du_1 \int_0^\infty u_2^{\sigma_2 - 1} e^{-s_2 u_2} S_n^{\alpha, \beta, 0} [u_2]\, du_2 \quad \ldots (2.6)$$

On expressing the general class of polynomials $S_V^U [u_1]$ and generalized polynomial set $S_n^{\alpha, \beta, 0} [\mu_2]$ occurring on the right hand side of (2.6) in series form with the help of (1.2) and (1.7) respectively, interchanging the order of integrals and summation in the result thus obtained and evaluating the u_1 and u_2 integrals with the help of a known formula [5,p.317,eqn.(3.381.4)], we arrive at the desired result.

3. SECOND INTEGRAL

$$\int_0^\infty\int_0^\infty (\lambda_1 x_1+\lambda_2 x_2)^{\sigma_1-1}(\mu_1 x_1+\mu_2 x_2)^{\sigma_2-1}$$
$$\exp[-s_1(\lambda_1 x_1+\lambda_2 x_2)-s_2(\mu_1 x_1+\mu_2 x_2)]\, S_V^U[\lambda_1 x_1+\lambda_2 x_2]$$
$$S_n^{\alpha,\beta,0}[\mu_1 x_1+\mu_2 x_2]\, H^{O,N:M_1,N_1:M_2,N_2}_{P,Q:\ P_1,Q_1\ :P_2,Q_2}\left[\begin{array}{c|l} z_1(\lambda_1 x_1+\lambda_2 x_2)^{\rho_1} & (a_j:\alpha_j',\alpha_j'')_{1,P}:(c_j',\gamma_j')_{1,P_1}:(c_j'',\gamma_j'')_{1,P_2} \\ z_2(\mu_1 x_1+\mu_2 x_2)^{\rho_2} & (b_j:\beta_j',\beta_j'')_{1,Q}:(d_j',\delta_j')_{1,Q_1}:(d_j'',\delta_j'')_{1,Q_2}\end{array}\right] dx_1\,dx_2$$

$$=\frac{1}{R}\sum_{K=0}^{[V/U]}\sum_{v=0}^{m+n}\sum_{u=0}^{v}\frac{\theta(K,u,v)}{s_1^{\sigma_1+K}\,s_2^{\sigma_2+qn+l(m+n)+vr}}\,H^{O,N:M_1,N_1+1:M_2,N_2+1}_{P,Q:P_1+1,Q_1:P_2+1,Q_2}$$
$$\left[\begin{array}{c|l} z_1 s_1^{-\rho_1} & (a_j:\alpha_j',\alpha_j'')_{1,P}:(1-\sigma_1-K,\rho_1),(c_j',\gamma_j')_{1,P_1}; \\ z_2 s_2^{-\rho_2} & (b_j:\beta_j',\beta_j'')_{1,Q}:(d_j',\delta_j')_{1,Q_1}; \end{array}\right.$$
$$\left.\begin{array}{c}(1-\sigma_2-qn-l(m+n)-vr,\rho_2),(c_j'',\gamma_j'')_{1,P_2} \\ (d_j'',\delta_j'')_{1,Q_2}\end{array}\right] \quad \text{... (3.1)}$$

where θ and R are defined as in (2.2) and (2.3) respectively and the following conditions are satisfied :

(i) $\rho_i \geq 0$; $\text{Re}(s_1) > 0$, $(i = 1, 2)$

(ii) $\text{Re}(\sigma_i) + \rho_i \min\limits_{1\leq j\leq M_i} \text{Re}\,(d_j^i/\delta_j^i) > 0 \quad (i = 1, 2)$

(iii) The H-function of two variables occurring in (3.1) satisfies the conditions corresponding appropriately to those given in the book [6,p.83,eqns.(6.1.5)-(6.1.6)]. ... (3.2)

Proof : To prove (3.1) we first express the H-function of two varaibels in terms of its Mellin-Barnes contour integrals with the help of (1.8) and interchange the order of ξ, η and x_1, x_2 integrals (which is permissible under the conditions stated with the integral (3.1)), to get

$$\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi(\xi,\eta)\,\phi_1(\xi)\,\phi_2(\eta)\,z_1^\xi z_2^\eta\,d\xi\,d\eta$$
$$\int_0^\infty\int_0^\infty(\lambda_1 x_1+\lambda_2' x_2)^{\sigma_1+\rho_1\xi-1}(\mu_1 x_1+\mu_2 x_2)^{\sigma_2+\rho_2\eta-1}$$

$$\exp\left[-s_1(\lambda_1 x_1+\lambda_2 x_2)-s_2(\mu_1 x_1+\mu_2 x_2)\right] S_V^U\left[\lambda_1 x_1+\lambda_2 x_2\right]$$

$$S_n^{\alpha,\beta,0}\left[\mu_1 x_1+\mu_2 x_2\right] dx_1\, dx_2 \qquad \ldots (3.3)$$

Now evaluating the above x_1, x_2 double integral with the help of (2.1) and reinterpreting the result, thus obtained in terms of the H-function of two variables, we easily arrive at the desired result after a little simplification.

4. SPECIAL CASES

(i) If we take $\lambda_1=\mu_2=1,\ \lambda_2=\mu_1=0$ in (3.1) and reduce the polynomial $S_V^U[x_1]$ in terms of Gould and Hopper polynomials $g_V^U[x_1, h]$ [4,p.58,eq.(6.2)] for which [see 14, p.161, eq.(1.15)] and the generalized polynomials set $S_n^{\alpha,\beta,0}[x_2]$ in terms of Gould and Hopper polynomials $H_n^{(r)}[x_2, \alpha, \beta]$ [u, p.52] for which [see, 9, p.65], we arrive at the following double Laplace transform :

$$L\left\{x_1^{\sigma_1-1} x_2^{\sigma_2-1} g_V^U[x_1, h]\, H_n^{(r)}[x_2, \alpha, \beta]\, H[z_1 x_1^{\rho_1}, z_2 x_2^{\rho_2}] : s_1, s_2\right\}$$

$$= \sum_{K=0}^{[V/U]} \sum_{v=0}^{n} \sum_{u=0}^{v} \frac{(UK)!\binom{V}{UK} h^K (-1)^n (-v)_u (-\alpha-ur)_n \beta^v}{K!\, u!\, v!\, s_1^{\sigma_1'} s_2^{\sigma_2-n+vr}}$$

$$H^{O,N:M_1,N_1+1:M_2,N_2+1}_{P,Q:P_1+1,Q_1:P_2+1,Q_2}\left[\begin{matrix} z_1 s_1^{-\rho_1} \\ z_2 s_2^{-\rho_2}\end{matrix}\middle| \begin{matrix}(a_j:\alpha_j',\alpha_j'')_{1,P}:(1-\sigma_1',\rho_1),(c_j',\gamma_j')_{1,P_1}; \\ (b_j:\beta_j',\beta_j'')_{1,Q}:(d_j',\delta_j')_{1,Q_1};\end{matrix}\right.$$

$$\left.\begin{matrix}(1-\sigma_2+n-vr,\rho_2),(c_j'',\gamma_j'')_{1,P_2} \\ (d_j'',\delta_j'')_{1,Q_2}\end{matrix}\right] \qquad \ldots (4.1)$$

where $\qquad \sigma_1' = \sigma_1 + (V-UK) \qquad \ldots (4.2)$

and the conditions easily obtainable from those stated with (3.1) are satisfied.

(ii) If we take $\lambda_1=\mu_2=1,\ \lambda_2=\mu_1=0$ in (3.1) and reduce the H-function of two variables in terms of product of a Whittaker and a modified Bessel's function [13, p.90, eq. (6.4.15), p.18, eqs. (2.6.7), (2.6.6)], the polynomials $S_V^U[x_1]$ in terms of Hermite polynomials $H_n(1/\sqrt{x_1})$ with the help of [14,p.158,eq.(1.4)] and the polynomial set $S_n^{\alpha,\beta,0}[x_2]$ in terms of Laguerre polynomials $L_n^{(\alpha)}[x_2]$ by taking $\beta=r=1$ in [9, p.65, eq. (2.1.15)], we arrive at a result given by Gupta and Agrawal [6, p.303, eq. (3.2)].

(iii) If in (3.1), we reduce the polynomials $S_V^U[\lambda_1 x_1+\lambda_2 x_2]$ to Laguerre polynomials $L_V^{(p)}[\lambda_1 x_1+\lambda_2 x_2]$ using [14, p.159, eq.11.8] and the generalized polynomial set $s_n^{\alpha,\beta,0}[\mu_1 x_1+\mu_2 x_2]$ to polynomials $L_n^{(\alpha)}[\mu_1 x_1+\mu_2 x_2, r, \beta]$ introduced by Singh and Srivastava [11] by using [9, p.65, eq. (2.1.15)], we arrive at the following integral

$$\int_0^\infty \int_0^\infty (\lambda_1 x_1+\lambda_2 x_2)^{\sigma_1-1}(\mu_1 x_1+\mu_2 x_2)^{\sigma_2-1}$$
$$\exp[-s_1(\lambda_1 x_1+\lambda_2 x_2)-s_2(\mu_1 x_1+\mu_2 x_2)]$$
$$L_V^{(p)}[\lambda_1 x_1+\lambda_2 x_2]\, L_n^{(\alpha)}[\mu_1 x_1+\mu_2 x_2, r, \beta]$$
$$H\left[\begin{matrix} z_1(\lambda_1 x_1+\lambda_2 x_2)^{\rho_1} \\ z_2(\mu_1 x_1+\mu_2 x_2)^{\rho_2}\end{matrix}\right] dx_1\, dx_2$$
$$=\frac{1}{R}\sum_{K=0}^{V}\sum_{v=0}^{n}\sum_{u=0}^{v}\frac{(-V)_k(-v)_u\,\beta^v(-\alpha-n-ru)_n}{K!\,u!\,v!\,(1+p)_K}\binom{V+p}{V}$$
$$H^{O,N:M_1,N_1+1:M_2,N_2+1}_{P,Q:P_1+1,Q_1\;:P_2+1,Q_2}\left[\begin{matrix} z_1 s_1^{-\rho_1} \\ z_2 s_2^{-\rho_2}\end{matrix}\middle| \begin{matrix}(a_j:\alpha_j',\alpha_j'')_{1,P}:(1-\sigma_1-K,\rho_1),(c_j',\gamma_j')_{1,P_1};(1-\sigma_2-vr,\rho_2),(c_j'',\gamma_j'')_{1,P_2} \\ (b_j:\beta_j',\beta_j'')_{1,Q}:(d_j',\delta_j')_{1,Q_1};(d_j'',\delta_j'')_{1,Q_2}\end{matrix}\right] \quad \ldots (4.3)$$

where R is defined as in (2.3) and the conditions easily derivable from (3.1) are satisfied.

Further, if we take $\sigma_1=\sigma_2=1=s_1=s_2$ in (4.3), reduce the polynomials $L_n^{(\alpha)}[\mu_1 x_1+\mu_2 x_2, r, \beta]$ to Laguerre polynomials $L_n^{(\alpha)}[\mu_1 x_1+\mu_2 x_2]$ by taking $\beta=r=1$ and the H-function of two variables into unity [By first reducing the H-function of two variables into product of two Fox's H-functions and then each of Fox's H-functions into exponential function with the help of results [13,p.90, eq.(6.4.15) and p.18, eq.(2.6.2)]. Next taking the limit as $z_1, z_2 \to 0$ and suitably specializing the parameters], we arrive at a known result given by Dhawan [2, p.417, eq. (2.2)] after a little simplification.

A number of other results involving products of several simpler functions and simple polynomials, obtained as special cases of H-function of two variables and the general polynomials $S_U^V[x]$ and the polynomial set $s_n^{\alpha,\beta,0}[x]$ can be obtained as special cases of our main result (3.1) but we do not record them here on account of lack of space.

REFERENCES

[1] S.K. Chatterjea, Quelques functions géněratrices des polynômes d'Hermite, du poiunt de vue de l'algebre de Lie, *C.R. Acad. Sci. Paris Ser. A-B* **268** (1969), A600-A602.

[2] G.K. Dhawan, On integrals involving classical polynomials, *Proc. Camb. Phil. Soc.* **64** (1968), 417.

[3] S.S. Dhillon, *A study of Generalization of Special Functions of Mathematical Physics and Applications,* Ph.D. Thesis, Bundelkhand Univ., India, 1989.

[4] H.W. Gould and A.T. Hopper, Operational formulas connected with two generalizations of Hermite polynomials, *Duke Math. J.* 29 (1962), 51-63.

[5] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, Inc. New York, 1980.

[6] K.C. Gupta and Pawan Agrawal, A study of unified double and multivariable integrals and the Laplace transforms associated with them, *Bull. Cal. Math. Soc.* **87** (1995), 299-306.

[7] H.L. Krall and O. Frink, A new class of orthogonal polynomials : The Bessel polynomials, *Trans. Amer. Math. Soc.* **65** (1949), 100- 115.

[8] P.K. Mittal and K.C. Gupta, An integral involving generalized function of two variables, *Proc. Indian Acad. Sci.* **78**(A) (1972), 117-123.

[9] S.K. Raizada, *A Study of Unified Representation of Special Functions of Mathematical Physics and Their Use in Statistical and Boundary Value Problems*, Ph.D. Thesis, Bundelkhand Univ., India, 1991.

[10] A. Singh, *A Study of Special Functions of Mathematical Physics and Their Applications in Combinateral Analysis*, Ph.D. Thesis, Bundelkhand Univ., India, 1981.

[11] R.P. Singh and K.N. Srivastava, A note on generalization of Laguerre and Humbert polynomials, *Ricerco (Napoli)* (2) **14** (1963), 11-21; Errata, ibid. **15** (1964), 63.

[12] H.M. Srivastava, A contour integral involving Fox's H-function, *Indian J. Math.* **14** (1972), 1-6.

[13] H.M. Srivastava, K.C. Gupta and S.P. Goyal, *The H-Functions of One and Two Variables with Applications*, South Asian Publishers, New Delhi and Madras, 1982.

[14] H.M. Srivastava and N.P. Singh, The integration of certain products of multivariable H-function with a general class of polynomials, *Rend. Circ. Mat. Palermo* **32**(2) (1983), 157-187.

[15] D.V. Widder, *Advanced Calculus*, Dover Publications, Inc. New York, 1968.