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SPECIAL, UNION AND HYPERASYMPTOTIC CURVES OF A TACHIBANA HYPERSURFACE

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ABSTRACT

Special, union and hyperasymptotic curves of a Riemannian hypersurface have been studied by Singh [1]. The object of this paper is to be investigate these curves in a Tachibana hypersurface.

1. INTRODUCTION. In an $(n+1)$ dimensional complex space C_{n+1} referred to an allowable coordinate system

$$(z^i, z^{\bar{i}}) \equiv (z^1, z^2, \ldots, z^{n+1}, z^{\bar{1}}, z^{\bar{2}}, \ldots, z^{\overline{n+1}})^*,$$

let us introduce the metric defined by the positive Hermitian form

$$ds^2 = 2g_{i\bar{j}}(z^i, z^{\bar{i}})\, dz^i\, dz^{\bar{j}}\,. \qquad \ldots (1.1)$$

If the tensor $g_{i\bar{j}}$ also satisfies the Kaehler's condition

$$\frac{\partial g_{i\bar{j}}}{\partial z^{\bar{k}}} = \frac{\partial g_{i\bar{k}}}{\partial z^{\bar{j}}}\,, \qquad \ldots (1.2)$$

then the complex space with metric satisfying the condition (1.2) is called Kaehler space. We always assume the self adjointness of the indices [2].

If a Kaehler space satisfies the conditions

$$F^h_{j,i} = 0,\ F^{\bar{h}}_{\bar{j},\bar{i}} = 0, \qquad \ldots (1.3a)$$

$$F_i = -F^k_{i,j} = 0,\ F_{\bar{i}} = -F^{\bar{k}}_{\bar{i},\bar{j}} = 0. \qquad \ldots (1.3b)$$

where the comma (,) followed by an index denotes the covarient differentiation w.r.t. the Christoffel's symbols formed with metric tensor, then the space is called an $(n+1)$ dimensional Tachibana space and is denoted by T^c_{n+1}.

The angle θ between two self adjoint vectors $\bar{u}$ and $\bar{v}$, whose components are $(r^i, r^{\bar{i}})$ and $(s^i, s^{\bar{i}})$ is given by

$$\cos\theta = \frac{g_{i\bar{j}}\, r^i\, s^{\bar{j}} + g_{\bar{i}j}\, r^{\bar{i}}\, s^j}{2\sqrt{(g_{i\bar{j}}\, r^i\, r^{\bar{j}})}\,\sqrt{(g_{\bar{i}j}\, s^{\bar{i}}\, s^j)}} \qquad \dots (1.4)$$

Let us consider an analytic hypersurface T_n^c of T_{n+1}^c. If $(u^\alpha, u^{\bar{\alpha}}) \equiv (u^1, \dots, u^n, u^{\bar{1}}, \dots, u^{\bar{n}})$ denote the coordinates of a point in T_n^c, the equations of the analytic hypersurface T_n^c may be written in the form

$$z^i = z^i\,(u^\alpha),\ z^{\bar{i}} = z^{\bar{i}}\,(u^{\bar{\alpha}}) \qquad \dots (1.5)$$

We quote below some fundamental formulae from [3] which will be used in the later part of this paper. Suppose that $g_{\alpha\beta}$ is the fundamental metric tensor of T_n^c, then we have

$$g_{\alpha\bar{\beta}} = g_{i\bar{j}}\, B_\alpha^i\, B_{\bar{\beta}}^{\bar{j}} \qquad \dots (1.6)$$

where $B_\alpha^i = \dfrac{\partial z^i}{\partial u^\alpha}$, $B_{\bar{\beta}}^{\bar{i}} = \dfrac{\partial z^{\bar{i}}}{\partial u^{\bar{\beta}}}$.

If $(N^i, N^{\bar{i}})$ be the components of unit normal vector to the hypersurface, then

$$2 g_{i\bar{j}}\, N^i\, N^{\bar{j}} = 1 \qquad \dots (1.7)$$

and

$$g_{i\bar{j}}\, N^i\, B_{\bar{\beta}}^{\bar{j}} = 0,\ g_{i\bar{j}} N^{\bar{j}}\, B_\alpha^i = 0\,. \qquad \dots (1.8)$$

The unit vector $(\xi^i, \xi^{\bar{i}})$ orthogonal to $\left(\dfrac{dz^i}{ds}, \dfrac{dz^{\bar{i}}}{ds}\right)$ is given by

$$g_{i\bar{j}}\, \frac{dz^i}{ds}\, \xi^{\bar{j}} + g_{\bar{i}j}\, \frac{dz^{\bar{i}}}{ds}\, \xi^j = 0 \qquad \dots (1.9)$$

and

$$2\, g_{i\bar{j}}\, \xi^i\, \xi^{\bar{j}} = 1\,.$$

Consider a curve $C : z^i = z^i(s)$, $z^{\bar{i}} = z^{\bar{i}}\,(s)$ (where s is real) of T_n^c. The components $\dfrac{dz^i}{ds}\left(\text{or } \dfrac{dz^{\bar{i}}}{ds}\right)$ and $\dfrac{du^\alpha}{ds}\left(\text{or } \dfrac{du^{\bar{\alpha}}}{ds}\right)$ of the unit tangent vectors of C with respect to the enveloping space and the hypersurface are related by

$$\frac{dz^i}{ds} = B_\alpha^i\, \frac{du^\alpha}{ds}\,, \qquad \dots (1.11a)$$

$$\frac{dz^{\bar{i}}}{ds} = B_{\bar{\alpha}}^{\bar{i}}\, \frac{du^{\bar{\alpha}}}{ds} \qquad \dots (1.11b)$$

If $(q^i, q^{\bar{i}})$ and $(p^\alpha, p^{\bar{\alpha}})$ are the components of the first curvature vectors with respect to T^c_{n+1} and T^c_n respectively, then

$$q^i = B^i_\alpha p^\alpha + K_n N^i \quad \text{... (1.12a)}$$

$$q^{\bar{i}} = B^{\bar{i}}_{\bar{\alpha}} p^{\bar{\alpha}} + \bar{K}_n N^{\bar{i}} . \quad \text{... (1.12b)}$$

where K_n is the component of the normal curvature of T^c_n in the direction of the curve C and $\bar{K}_n$ is its complex conjugate, and

$$\left.\begin{array}{lll} q^i = dz^i/ds, & \beta\, du^\beta/ds + dz^i/ds, & \bar{\beta}\, du^\beta/ds \\ q^{\bar{i}} = dz^{\bar{i}}/ds, & \beta\, du^\beta/ds + dz^{\bar{i}}/ds, & \bar{\beta}\, du^\beta/ds \\ p^\alpha = du^\alpha/ds, & \beta\, du^\beta/ds + du^\alpha/ds, & \bar{\beta}\, du^\beta/ds \\ p^{\bar{\alpha}} = du^{\bar{\alpha}}/ds, & \beta\, du^\beta/ds + du^{\bar{\alpha}}/ds, & \bar{\beta}\, du^\beta/ds \end{array}\right\} \quad \text{... (1.13)}$$

$$B^i_{\alpha,\beta} = \Omega_{\alpha\beta} N^i, \qquad B^{\bar{i}}_{\bar{\alpha},\bar{\beta}} = \Omega_{\bar{\alpha}\bar{\beta}} N^{\bar{i}}$$

$$K_n = \Omega_{\alpha\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds}, \qquad \bar{K}_n = \Omega_{\bar{\alpha}\bar{\beta}} \frac{du^{\bar{\alpha}}}{ds} \frac{du^{\bar{\beta}}}{ds}.$$

$\Omega_{\alpha\beta}$ and $\Omega_{\bar{\alpha}\bar{\beta}}$ are the components of the second fundamental tensors of the hypersurface, $(q^i, q^{\bar{i}})$ and $(p^\alpha, p^{\bar{\alpha}})$ are the components of the first curvature vector of the curve with respect to T^c_{n+1} and T^c_n respectively.

2. Union and Special Curves in T^c_n. The first two of Frenet's formulae of a curve in T^c_{n+1} are given by

$$\frac{\delta \eta^i_{(0)}}{\delta s} = K_{(1)} \eta^i_{(1)} \text{ and } \frac{\delta \eta^i_{(1)}}{\delta s} = -K_{(1)} \eta^i_{(0)} + K_{(2)} \eta^i_{(2)} \quad \text{... (2.1)}$$

and their conjugates, where

$$(\eta^i_{(0)}, \eta^{\bar{i}}_{(0)}) = (dz^i/ds, dz^{\bar{i}}/ds),\ (\eta^i_{(1)}, \eta^{\bar{i}}_{(1)}) \text{ and } (\eta^i_{(2)}, \eta^{\bar{i}}_{(2)})$$

are the components of unit tangent, unit principal normal vector and unit first binormal vector, $K_{(1)}$ and $K_{(2)}$ are the first and second curvatures of the curve in the embedding space, $\delta/\delta s$ is the usual covariant differentiation.

Consider two congruences of curves $\bar{\lambda}$ and $\bar{\mu}$ given at any point of T^c_n by

$$\lambda^i = \gamma^\alpha B^i_\alpha + cN^i \quad \text{... (2.2a)}$$

its conjugate,

$$\mu^i = s^\alpha B^i_\alpha + DN^i \quad \text{... (2.2b)}$$

and its conjugate.

Let Γ be a special curve relative to the congruence $\overline{\lambda}$ and an union curve relative to the congruence $\overline{\mu}$. A curve of the congruence is said to be a special curve relative to the congruence $\overline{\lambda}$ if the vector $(\lambda^i, \lambda^{\bar{i}})$ lies in a variety spanned by the vectors $(p^\alpha B^i_\alpha, p^{\bar{\alpha}} B^{\bar{i}}_{\bar{\alpha}})$ and $(q^i, q^{\bar{i}})$ and a curve of the congruence is said to be union curve relative to the congruence $\overline{\mu}$ if the vector $(\mu^i, \mu^{\bar{i}})$ lies in a variety spanned by the vectors $(dz^i/ds, dz^{\bar{i}}/ds)$ and $(q^i, q^{\bar{i}})$. In other words

$$(\lambda^i, \lambda^{\bar{i}}) = u(p^\alpha B^i_\alpha, p^{\bar{\alpha}} B^{\bar{i}}_{\bar{\alpha}}) + w(q^i, q^{\bar{i}})$$

and $$(\mu^i, \mu^{\bar{i}}) = x(dz^i/ds, dz^{\bar{i}}/ds) + y(q^i, q^{\bar{i}})$$

which yield

$$\lambda^i = up^\alpha B^i_\alpha + wq^i \text{ and its conjugate,} \quad \text{... (2.3a)}$$

$$\mu^i = x\, dz^i/ds + yq^i \text{ and its conjugate.} \quad \text{... (2.3b)}$$

From equations (1.12a), (2.2a), (2.3a), (2.3b) and their conjugates one obtains

$$r^\alpha = (u + w)\, p^\alpha,\ C = wK_n \quad \text{... (2.4)}$$

$$s^\alpha = x\, dx^\alpha/ds + yp^\alpha,\ D = yK_n \quad \text{... (2.5)}$$

and their conjugate relations.

We define

$$R^2 = 2g_{\alpha\bar{\beta}}\, \gamma^\alpha \gamma^{\bar{\beta}},\ S^2 = 2g_{\alpha\bar{\beta}}\, s^\alpha s^{\bar{\beta}},\ k^2_{(1)} = 2g_{\alpha\bar{\beta}}\, p^\alpha p^{\bar{\beta}} \quad \text{... (2.6)}$$

where $k_{(1)}$ is the first curvature of the curve in T^c_n. Using (1.9), (2.4), its conjugate, (2.5) and its conjugate, we obtain from these equations

$$K_n S \cos\theta = Dk_{(1)} \quad \text{... (2.7)}$$

and its conjugate.

From this equation and the equations

$$1 = 2 g_{i\bar{j}}\, \mu^i \mu^{\bar{j}} = S^2 + D\overline{D} \quad \text{... (2.8)}$$

$$K^2_{(1)} = 2g_{i\bar{j}}\, q^i q^{\bar{j}} \quad \text{... (2.9)}$$

where $$K^2_{(1)} = k^2_{(1)} + K_n \overline{K}_n$$

We obtain

$$|K_n| = \{eK_{(1)} (1 - S^2)^{1/2}\}/(1 - S^2 \sin^2\theta)^{1/2} \quad \text{... (2.10)}$$

and $$k_{(1)} = \{eK_{(1)}\, S \cos\theta\}/(1 - S^2 \sin^2\theta)^{1/2} \quad \text{... (2.11)}$$

where $e = \pm 1$ in order that $e \cos\theta$ be non-negative.

Theorem 2.1 . If a special curve relative to a fixed congruence $\overline{\lambda}$ is an union curve relative to another fixed congruence $\overline{\mu}$, then the modulus of the normal and the first curvatures with respect to T_n^c at a given point of the curve are proportional to its first curvature with respect to T_{n+1}^c.

Theorem 2.2 . If the components of the vector fields $\overline{\lambda}$ and $\overline{\mu}$ tangent to the hypersurface are in the same direction, then the ratio of the two first curvatures is equal to the magnitude of the tangential (to the hypersurface) component of $\overline{\mu}$.

Proof . $\theta = 0$ proves both the theorems.

3. HYPERASYMPTOTIC CURVES IN T_n^c . A curve of the hypersurface is said to be an hyperasymptotic curve relative to the congruence $\overline{\mu}$, if the vector $(\mu^i, \mu^{\bar{i}})$ lies in the variety spanned by the vectors $(\eta^i_{(0)}, \eta^{\bar{i}}_{(0)})$ and $(\eta^i_{(2)}, \eta^{\bar{i}}_{(2)})$.

In other words

$$(\mu^i, \mu^{\bar{i}}) = X_{(1)}(\eta^i_{(0)}, \eta^{\bar{i}}_{(0)}) + Y_{(1)}(\eta^i_{(2)}, \eta^{\bar{i}}_{(2)})$$

which yields

$$\mu^i = X_{(1)}\eta^i_{(0)} + Y_{(1)}\eta^i_{(2)} \text{ and its conjugate.} \qquad \dots (3.1)$$

From the equation (2.1) and its conjugate we deduce

$$\frac{\delta q^i}{\delta s} = -K^2_{(1)}\eta^i_{(1)} + \frac{d}{ds}\log K_{(1)}\, q^i + K_{(1)}K_{(2)}\eta^i_{(2)} \qquad \dots (3.2)$$

and its conjugate.

Another expression for $\left(\frac{\delta q^i}{\delta s}, \frac{\delta q^{\bar{i}}}{\delta s}\right)$ will be obtained from the equation (1.12 a, b) after using

$$\frac{\delta N^i}{\delta s} = -\frac{1}{2}\Omega_{\beta\bar{\gamma}}g^{\beta\alpha}B^i_\alpha\frac{dx^{\bar{\gamma}}}{ds} \text{ and its conjugate.} \qquad \dots (3.3)$$

Equations (1.12a), (2.1), (3.1), (3.2) and their conjugates yield

$$s^\alpha = X_{(1)}\frac{dx^\alpha}{ds} + z\left(\frac{\delta p^\alpha}{\delta s} - K_{n/2}\Omega_{\beta\bar{\gamma}}g^{\bar{\beta}\alpha}\frac{dx^{\bar{\gamma}}}{ds} - p^\alpha\frac{d}{ds}\log K_{(1)} + K^2_{(1)}\frac{dx^\alpha}{ds}\right) \qquad \dots (3.4)$$

$$D = z\left(\Omega_{\alpha\beta}\, p^{\alpha} \frac{dx^{\beta}}{ds} + \frac{d}{ds} K_n - K_n \frac{d}{ds} \log k_{(1)} \right) \qquad \ldots (3.5)$$

and their conjugates, where $z = Y_{(1)}/K_{(1)}\, K_{(2)}$.

Let $(\xi^{\alpha}_{(0)}, \xi^{\bar{\alpha}}_{(0)}, \xi^{\alpha}_{(1)}, \xi^{\bar{\alpha}}_{(1)})$ and $(\xi^{\alpha}_{(2)}, \xi^{\bar{\alpha}}_{(2)})$ be the unit tangent, unit principal, normal, unit first binormal vectors, $k_{(1)}$ and $k_{(2)}$ be the first and second curvatures of the curve with respect to the hypersurface. We obtain from Frenet's formulae with respect to T^c_n.

$$\frac{\delta p^{\alpha}}{\delta s} = -k^2_{(1)} \frac{dx^{\alpha}}{ds} + p^{\alpha} \frac{d}{ds} \log k_{(1)} + k_{(1)} k_{(2)}\, \xi^{\alpha}_{(2)} \qquad \ldots (3.6)$$

and its conjugate.

Equations (3.4), (3.5), (3.6) their conjugates and the definition

$$\cos\phi \left\{ q_{\alpha\bar{\beta}}\, s^{\alpha} \frac{dx^{\bar{\beta}}}{ds} + g_{\bar{\alpha}\beta}\, s^{\bar{\alpha}} \frac{dx^{\beta}}{ds} \right\} \quad S$$

yield

$$S \cos\phi = X(1) \qquad \ldots (3.7)$$

and
$$\left(s^{\alpha} - S\cos\phi \frac{dx^{\alpha}}{ds} \right) \left[\Omega_{\gamma\beta}\, p^{\gamma} \frac{dx^{\beta}}{ds} + \frac{d}{ds} K_n - K_n \frac{d}{ds} \log k_{(2)} \right]$$

$$= D\left[K_n \bar{K}_n \frac{dX^{\alpha}}{ds} + p^{\alpha} \frac{d}{ds} \log \frac{k_{(1)}}{K_{(1)}} + k_{(1)}\, k_{(2)}\, \xi^{\alpha}_{(2)} - K_n / 2\, \Omega_{\bar{\beta}\bar{\gamma}}\, g^{\bar{\alpha}\beta} \frac{dx^{\bar{\gamma}}}{ds} \right]$$

and its conjugate.

A hyperasymptotic curve relative to $\bar{\mu}$ is characterised by this equation.

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