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EFFECT OF SUSPENDED PARTICLES ON THERMAL INSTABILITY OF A ROTATING MAXWELLIAN VISCOELATIC FLUID IN POROUS MEDIUM

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ABSTRACT

The thermal instability of a rotating Maxwellian viscoelastic fluid permeated with suspended partiles, in porous medium, is considered. The presence of each-rotation, suspended particles and viscoelasticity brings oscillatory modes in the system which were nonexistent in their absence. For stationary convection, the Maxwellian viscoelastic fluid behaves like a Newtonian fluid and the medium permeability succeeds in stabilizing the thermal instability of a fluid-particle layer for certain wave numbers in the presence of rotation which were unstable in the absence of rotation. However, the suspended particles and the rotation are found to have destabilizing and stabilizing effects respectively on the thermal instability problem for stationary convection.

1. INTRODUCTION. A detailed account of the thermal instability in Newtonian fluids has been given by Chandrasekhar [1]. Lapwood [2] has studied the stability of convective flow in a porous medium using Rayleigh's procedure. The Rayleigh instability of a thermal boundary layer in flow through a porous medium has been considered by Wooding [3]. When the fluid slowly percolates through the pores of the rock, the gross effect is represented by the well-known Darcy's law.

Chandra [4] observed in an air layer that convection occured at much lower gradients than predicted and appeared as irregular strips of elongated cells with fluid rising at the centre if the layer depth was less than 7 mm and called this motion "columnar instability". However, for layers deeper than 10 mm, a Bénard type cellular convection was observed. Chandra [4] added an aerosol to mark the flow pattern. Thus there is a decades-old contradiction between the theory and the experiment. Scanlon and Segel [5] have considered the effect of suspended particles on the onset of thermal convection and found that the critical Rayleigh number was reduced solely because the heat capacity of the pure fluid was supplemented by that of the particles. The effect of suspended particles was thus found to destablize the layer.

In all the above studies, the fluid has been considered to be Newtonian. The stability of a horizontal layer of Maxwell's viscoelastic fluid heated from below has been investigated by Vest and Arpaci [6]. The nature of the instability and some factors may have different effects on viscoelastic fluids as compared to the Newtonian fluids. For example, Bhatia and Steiner [7] have considered the effect of a uniform rotation on the thermal instability of a Maxwell fluid and have found that rotation has a destabilizing effect in contrast to the stabilizing effect on Newtonian fluid. Experimental demonstration by Toms and strawbridge [8] has revealed that a dilute solution of methyl methacrylate in n-butyl acetate agrees well with the the theoretical model of Oldroyd fluid. The thermal instability of an Oldroydian viscoelastic fluid has been considered in the presence of rotation (Eltayeb [9], Sharma [10]).

The present paper attempts to study the thermal instability of rotating Maxwellian viscoelastic fluid, permeated with suspended particles, in porous medium. The knowledge regarding fluid- particle mixtures is not commensurate with their scientific and industrial importance. The analysis would be relevant to the stability of some polymer solutions and the problem finds its usefulness in several geophysical situations (e.g. convection in petroleum deposits below the surface of Earth where the Earth's rotation plays a significant role on convection and in extraction of energy) and in chemical technology. These aspects form the motivation for the present study.

2. PERTURBATION EQUATIONS. Consider an infinite horizontal layer of a Maxwellian viscoelastic fluid permeated with suspended particles and bounded by the planes $z = 0$ and $z = d$ in porous medium. This layer is heated from below such that a steady adverse temperature gradient $\beta(= |\frac{dT}{dz}|)$ is maintained. The system is acted on by gravity force $\boldsymbol{g}(0, 0, -g)$ and uniform rotation $\vec{\Omega}(0, 0, \Omega)$. Let T_{ij}, τ_{ij}, e_{ij}, δ_{ij}, μ, λ, p, u_i, x_i and $\frac{d}{dt}$ denote respectively the total stress tensor, the shear stress tensor, the rate-of-strain tensor, the Kronecker delta, the viscosity, the stress relaxation time, the isotropic pressure, the velocity vector, the position vector and the mobile operator. Then the Maxwellian viscoelastic fluid is described by the constitutive relations

$$T_{ij} = -p\delta_{ij} + z_{ij},$$

$$(1 + \lambda\frac{d}{dt})\, z_{ij} = 2\mu e_{ij},$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad ...(1)$$

When the fluid slowly percolates through the pores of the rock, the gross effect is represesneted by the usual Darcy's law. As a consequence, the resistance term $-(\frac{\mu}{k_1})v$ will replace the usual viscous term in the equations of motion. Here k_1 is the permeability of the medium and v is the filter velocity of the fluid, the porous medium being homogeneous and isotropic.

Let $\rho, \mu, p, \vec{u}(u, v, w)$ and $\vec{\Omega}(0, 0, \Omega)$ denote respectively, the density, the viscosity, the pressure, the velocity of pure fluid and uniform rotation; $v(\bar{x}, t)$ and $N(\bar{x}, t)$ denote the velocity and the number density of the particles respecitvely. $k = 6\pi\rho\upsilon\eta$, where η is the particle radius is the Stoke's drag coefficient $v = (l, r, s)$, $\bar{x} = (x, y, z)$, $\bar{\lambda} = (0, 0, 1)$ and ε is the medium porosity. Then the equations of motion and continuity for the rotating Maxwellian viscoelastic fluid permeated with suspended particles in a porous medium are

$$\left(1 + \lambda\frac{d}{dt}\right)\frac{\rho}{\varepsilon}\left[\frac{\partial \boldsymbol{u}}{\partial t} + \frac{1}{\varepsilon}(\vec{\boldsymbol{u}}.\nabla)\vec{\boldsymbol{u}}\right] = \left(1 + \lambda\frac{d}{dt}\right)\left[-\nabla p - \rho g\bar{\lambda} + \frac{kN}{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}) + \frac{2\rho}{\varepsilon}(\boldsymbol{u} \times \vec{\Omega})\right] - \frac{\mu}{k_1}\boldsymbol{u} \qquad ...(2)$$

$$\nabla \,.\, \boldsymbol{u} = 0. \qquad ...(3)$$

In the equations of motion (2), the presence of partilces adds an extra force term, proportional to the velocity difference between particles and fluid. Since the force exerted by the fluid on the particles is equal and opposite to that exerted by the particles on the fluid, there must be an extra force term, equal in magnitude but opposite in sign, in the equations of motion of the particles. The distances between particles are assumed quite large compared with their diameter so that interparticle reactions are ignored. The effects of pressure, gravity, Darcian force and Coriolis force on the suspended particles are negligibly small and therefore ignored. If mN is the mass of particles per unit volume, then the equations of motion and continuity for the particles, under the above assumptions are

$$mN\left[\frac{\partial \boldsymbol{v}}{\partial t}+\frac{1}{\varepsilon}(\boldsymbol{v}\,.\,\nabla)\boldsymbol{v}\right]=kN(\boldsymbol{u}-\boldsymbol{v}), \qquad ...(4)$$

$$\varepsilon\frac{\partial N}{\partial t}+\nabla.\,(N\boldsymbol{v})=0. \qquad ...(5)$$

Let c, c_{pt}, T and q denote respectively the heat capacity of the fluid, the heat capacity of the particles, the temperature and the "effective thermal conductivity" of the pure fluid. When the fluid and the particles are in thermal equilibrium, the equation of heat conduction gives

$$[\rho c\varepsilon+\rho_s c_s(1-\varepsilon)]\frac{\partial T}{\partial t}+\rho c(\boldsymbol{u}.\nabla)T+mNc_{pt}\left(\varepsilon\frac{\partial}{\partial t}+\boldsymbol{v}\,.\,\nabla\right)T$$

$$=q\nabla^2 T, \qquad ...(6)$$

where ρ_s, c_s stand for the density and the heat capacity of the solid matrix, respectively.

The initial state of the system is taken to be a quiescent layer (no settling) with a uniform particle distribution N_0. The initial state

$$\boldsymbol{u}=0,\ \boldsymbol{v}=0,\ T=-\beta z,\ N_0=\text{constant} \qquad ...(7)$$

is an exact solution to the governing equations. The equatuion of state for the fluid is

$$\rho=\rho_0\,[1-\alpha(T-T_0)], \qquad ...(8)$$

where ρ_0, T_0 are respectively, the mean density and the temperature of the clean fluid at the bottom surface $z=0$ and α is the coefficient of thermal expansion.

Let $\delta\rho$, δp, θ, $\boldsymbol{u}(u, v, w)$, $\boldsymbol{v}(l, r, s)$ and N denote respectively perturbations in density ρ, pressure p, temperature T, fluid velocity (zero initially), particle velocity (zero initially) and number denisty N_0. Then the linearized perturbation equations of the fluid-particle layer are

$$\left(1+\lambda\frac{d}{dt}\right)\frac{\rho_0}{\varepsilon}\frac{\partial \boldsymbol{u}}{\partial t}=\left(1+\lambda\frac{d}{dt}\right)\left[-\nabla\delta p+g\rho_0\alpha\theta\vec{\lambda}+\frac{kN_0}{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})\right.$$

$$+\frac{2\rho_0}{\varepsilon}(\boldsymbol{u}+\vec{\Omega})\Bigg]-\frac{\mu}{k_1}\boldsymbol{u} \quad \text{...(9)}$$

$$\nabla . \boldsymbol{u} = 0 \quad \text{...(10)}$$

$$mN_0\frac{\partial \boldsymbol{v}}{\partial t} = kn_0(\boldsymbol{u}-\boldsymbol{v}), \quad \text{...(11)}$$

$$\varepsilon\frac{\partial M}{\partial t} + \nabla.\boldsymbol{v} = 0 \quad \text{...(12)}$$

and

$$(E+h\varepsilon)\frac{\partial \theta}{\partial t} + (\boldsymbol{u}.\nabla)T + h(\boldsymbol{v}.\nabla)T = \frac{q}{\rho_0 c}\nabla^2\theta \quad \text{...(13)}$$

where $M = \frac{N}{N_0}, E = \varepsilon + (1-\varepsilon)\frac{\rho_s c_s}{\rho c}$,

$$h = \frac{mN_0}{\rho_0}\frac{c_{pt}}{c} = f\frac{c_{pt}}{c} \text{ and } f = \frac{mN_0}{\rho_0}.$$

Now introducing the non-dimensional quantities defined by

$$z = z^* d, t = \frac{d^2}{x}t^*, \boldsymbol{u} = \frac{x}{d}\boldsymbol{u}^*, p = \frac{\rho v x}{d^2}p^*, \theta = (\beta d)\theta^*, \Omega = \frac{x}{d^2}\Omega^* \quad \text{..(14)}$$

and omitting the asterisks for simplicity, (9)-(13) in the non-dimensional forms become

$$(1+\lambda'\frac{d}{dt})N_p^{-1}\frac{\partial \boldsymbol{u}}{\partial t} = (1+\lambda'\frac{d}{dt})\Bigg[-\nabla\delta p + N_R\theta\vec{\lambda} + \omega(\boldsymbol{v}-\boldsymbol{u})$$

$$+ 2N_p^{-1}(\boldsymbol{u}\times\vec{\Omega})\Bigg] - p^{-1}\boldsymbol{u}, \quad \text{...(15)}$$

$$\nabla.\boldsymbol{u} = 0, \quad \text{...(16)}$$

$$\left(z\frac{\partial}{\partial t}+1\right)\vec{v} = \vec{u}, \quad \text{...(17)}$$

$$(E+h\varepsilon)\frac{\partial \theta}{\partial t} = (w+hs) + \nabla^2\theta, \quad \text{...(18)}$$

where $N_p = \frac{\varepsilon v}{x}$ is the modified Prandtl number, $N_R = \frac{g\alpha\beta d^4}{\nu x}$ is the Rayleigh number, $\omega = \frac{kN_0 d^2}{\rho_0 v\varepsilon}$, $f = \frac{mN_0}{\rho_0} = zwN_p$ is the mass fraction, $z = \frac{mx}{kd^2}, p = \frac{k_1}{d^2}$ and $\lambda' = \frac{\lambda x}{d^2}$. Also w, s are the vertical fluid and particle velocities, v is the kinematic viscosity of the fluid and x is the thermal diffusivity.

Let us consider the case of two free surfaces having uniform temperatures. The scaled boundary conditions appropriate for the problem are

$$w = \frac{\partial^2 w}{\partial z^2} = \theta = 0 \text{ at } z = 0 \text{ and } 1. \quad \text{...(19)}$$

Eliminating $\boldsymbol{v}$ from (15), (17) and (18), the fluid and heat equations become

$$[L_1(1 + \lambda' \frac{d}{dt}) + p^{-1} L_2]\zeta = (1 + \lambda' \frac{d}{dt})(2L_2 N_p^{-1} \Omega D w), \quad \text{...(20)}$$

$$[L_1(1 + \lambda' \frac{d}{dt}) + p^{-1} L_2]\nabla^2 w + (1 + \lambda' \frac{d}{dt}) L_2 (N_R \nabla_1^2 \theta - 2N_p^{-1} \Omega D \zeta] \quad \text{...(21)}$$

and $$L_2[c(E + h\varepsilon) \frac{\partial}{dt} - \nabla^2]\theta = [z \frac{\partial}{\partial t} + H]w, \quad \text{...(22)}$$

where we have taken twice the curl of fluid equations and taking z-components only, and $\zeta = (\nabla \times \boldsymbol{u})_z$ is the z-components of vorticity

$$L_1 = N_p^{-1}\left(z \frac{\partial^2}{\partial t^2} + F \frac{\partial}{\partial t} \right), L_2 = z \frac{\partial}{\partial t} + 1, F = f + 1,$$

$$H = h + 1, \nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and $$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Following the normal mode analysis we assume that the solutions of (20)-(22) are given by

$$[w, \zeta, \theta] = [W(z), Z(z), \Theta(z)] \exp(ik_x x + ik_y y + nt), \quad \text{...(23)}$$

where n is the growth rate and $k = (k_x^2 + k_y^2)^{1/2}$ is the wave number of the disturbance.

Equations (20)-(22), with the help of expression (23), take the forms

$$[(1 + \lambda' n)L_1 + p^{-1} L_2]z = 2(1 + \lambda' n) L_2 N_p^{-1} \Omega D W, \quad \text{...(24)}$$

$$[(1 + \lambda' n)L_1 + p^{-1} L_2](D^2 - k^2)W = -(1 + \lambda' n) L_2 N_k k^2 \Theta$$
$$- 2(1 + \lambda' n) L_2 N_p^{-1} \Omega\, Dz \quad \text{...(25)}$$

$$L_2 [(E - h\varepsilon)n - (d^2 - k^2)\Theta = (zn + H)W \quad \text{...(26)}$$

Eliminating z and Θ between (24)-(26), we obtain

$$[(1+\lambda'n)L_1+p^{-1}L_2]^2\,[(E+h\varepsilon)n-(D^2-k^2)][(D^2-k^2)W$$
$$=-(1+\lambda'n)N_R k^2[(1+\lambda'n)L_1+p^{-1}L_2](zn+H)W$$
$$-4(1+\lambda'n)^2L_2^2N_p^{-2}\Omega^2\,[(E+h\varepsilon)n-(D^2-k^2)]D^2W \qquad ...(27)$$

where $L_1=N_p^{-1}(zn^2+F_n)$, $L_2=(1+zn)$ and $D=\frac{d}{dz}$.

Here we consider the case of two free boundaries. The dimensionless boundary conditions appropriate for the problem are

$$W=D^2W=Dz=\Theta=0 \text{ at } z=0 \text{ and } 1. \qquad ...(28)$$

The solution of (27) satisfying (28) is given by

$$W=W_0\sin\pi z, \qquad ...(29)$$

where W_0 is a contant. Substituting (29) in (27), we obtain the dispersion relations

$$N_R=\frac{[(E+h\varepsilon)n+(\pi^2+k^2)]\times[\{(1+\lambda'n)L_1+p^{-1}L_2\}^2(\pi^2+k^2)+4\pi^2L_2^2\Omega^2N_p^{-2}(1+\lambda'n)^2]}{k^2(1+\lambda'n)[(1+\lambda'n)L_1+p^{-1}L_2][zn+H]} \qquad ...(30)$$

3. THE STATIONARY CONVECTION. When the instability sets in as stationary convection, the marginal state will be characterized by $n=0$. Putting $n=0$, the dispersion relation (30) reduces to

$$N_R=\frac{(\pi^2+k^2)^2}{Hpk^2}+4N_p^{-2}\Omega^2\pi^2(\pi^2+k^2)\frac{p}{Hk^2} \qquad ...(31)$$

Thus for stationary convection, the stress relaxation time parameter vanishes with n and the Maxwell fluid behaves like a Newtonian fluid.

Equation (31) can be written as

$$N_R=\frac{(\pi^2+k^2)^2}{Hpk^2}+4\Omega^2\pi^2(\pi^2+k^2)\frac{p}{Hk^2N_p^2} \qquad ...(32)$$

To study the effects of suspended particles, medium permeability and rotation, we examine the natures of $\frac{dN_R}{dH}$, $\frac{dN_R}{dP}$ and $\frac{dN_R}{d\Omega}$ respectively. Equation (32) gives

$$\frac{dN_R}{dH} = -\left[\frac{(\pi^2 + k^2)^2}{H^2 k^2 p} + \frac{4\Omega^2\pi^2(\pi^2 + k^2)p}{H^2 k^2 N_p^2} \right], \qquad ...(33)$$

which is negative. The effect of suspended particles is, thus, destabilizing on thermal instability of fluid-particle layer in the presence of uniform rotation in porous medium.

From (32), it follows that

$$\frac{dN_R}{dp} = \left[\frac{(\pi^2 + k^2)^2}{Hk^2 p^2} - \frac{4\Omega^2\pi^2(\pi^2 + k^2)}{Hk^2 N_p^2} \right], \qquad ...(34)$$

which is, positive if

$$\left(\frac{\pi^2 + k^2}{p^2} \right) < \frac{4\Omega^2\pi^2}{N_p^2}$$

and is negative if

$$\left(\frac{\pi^2 + k^2}{p^2} \right) > \frac{4\Omega^2\pi^2}{N_p^2}.$$

Thus, the medium permeability has been stabilizing and destabilizing effects depending upon the values of the various parameters. In the absence of rotation, the medium permeability has a destabilizing effect since for this case

$$\frac{dN_R}{dp} = -\frac{(\pi^2 + k^2)^2}{Hp^2 k^2}, \qquad ...(35)$$

which is always negative. The medium permeability, thus, succeeds in stabilizing the thermal instability of fluid-particle layer for certain wave numbers in the presence of rotation, which were unstable in the absence of rotation.

Equation (32) also yields

$$\frac{dN_R}{d\Omega} = \frac{8\Omega\pi^2 (\pi^2 + k^2)p}{Hk^2 N_p^2} \qquad ...(36)$$

which is always positive. This shows that the rotation has a stabilizing effect on the system.

4. STABILITY OF THE SYSTEM AND OSCILLATORY MODES

Multiplying equation (25) by W^*, the complex conjugate of W, integrating over the range of z and making use of (24), (26) together with the boundary conditions (28), we obtain

$$L_2^*[zn^* + H][1 + \lambda'n^*][1 + \lambda'n)L_1 + p^{-1}L_2]I_1 = [(1 + \lambda'n^*)(1 + \lambda'n)$$
$$L_2L_2^{2*}N_Rk^2][(E + h\varepsilon)n^*I_2 + I_3] - L_2[zn^* + H][1 + \lambda'n]$$
$$[(1 + \lambda'n^*)L_1^* + p^{-1}L_2^*]I_4, \qquad ...(37)$$

where

$$I_1 = \int_0^1 (|DW|^2 + k^2|W|^2)\, dz,$$
$$I_2 = \int_0^1 |\Theta|^2\, dz,$$
$$I_3 = \int_0^1 (|D\Theta|^2 + k^2|\Theta|^2)\, dz,$$
$$I_3 = \int_0^1 |z|^2\, dz, \qquad ...(38)$$

which are all positive definite. Putting $n = in_0$, where n_0 is real, in (37) and equating the imaginary parts on both sides, we obtain

$$n_0\, [n_0^4\{N_p^{-1}z^2\lambda'^2(1 + H - F)\}I_1 + \{N_Rk^2(E + h\varepsilon)\lambda'^2z^2\}I_2$$
$$+ \{N_Rk^2\lambda'^2z^3\}I_3 + \{N_p^{-1}z^2\lambda'^2(1 - H - F)\}I_4]$$
$$+ n_0^2\, [\{H\lambda' + Hz + z) < N_p^{-1}\, (z + \lambda'F) > - (H\lambda' + \lambda' + z)$$
$$< N_p^{-1}\, Fz + z^2p^{-1} > - N_p^{-1}\lambda'zH + z^2\lambda'p^{-1}\}I_1$$
$$+ \{N_Rk^2\, (E + h\varepsilon)(z^2 + \lambda'^2)\}I_2 + \{N_Rk^2\, (z^3 + \lambda'^2z)\}I_3$$
$$+ \{N_p^{-1}z^2\, (1 - F - H) - N_p^{-1}\, HF\lambda'^2 + Hz^2\lambda'p^{-1} - z^3p^{-1}\}I_4]$$
$$+ \{N_Rk^2z\}I_3 + \{H\lambda'p^{-1} - N_p^{-1}HF - zp^{-1}\}\, I_4] = 0. \qquad ...(39)$$

Equation (39) implies that $n_0 = 0$ or $n_0 \neq 0$, which means that the modes may be oscillatory or non-oscillatory. In the absence of rotation, suspended particles and for Newtonian fluid ($\lambda = 0$), (39) reduces to

$$n_0\left[\frac{I_1}{N_p} + N_kk^2EI_2\right] = 0, \qquad ...(40)$$

and terms in brackets are positive definite. Thus $n_0 = 0$, which means that oscillatory modes are not allowed and the principle of exchange of stabilities is satisfied for a porous medium in the absence of rotation, suspended particles and for Newtonian fluid. This result is true for the porous as well as non-porous (Chandrasekhar [1]) medium. The presence of each-rotation, suspended particles and viscoelasticity, brings oscillatory modes (as n_0 may not be zero) which were non-existent in their absence.

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