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LAPLACE TRANSFORM OF GENERALIZED HYPERGEOMETRIC DISTRIBUTION FUNCTION ASSOCIATED WITH ZONAL POLYNOMIAL

By

P.L. Sethi and Pradhyot K.Mishra
Department of Mathematics and Statistics, Jai Narain Vyas University
Jodhpur - 342 001, Rajasthan, India

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ABSTRACT

In this paper authors first discuss probability density functions of generalized hypergeometric function of symmetric positive definite matrix associated with zonal polynomial, then establish new result, which has a wide range of application in mathematical Statistics by virtue of Laplace transform of matrix variate.

1. INTRODUCTION. At first Herz (1955) has introduced the theory of matrix argument involving certain special functions such as Bessel function, confluent hypergeometric function etc. Later on James (1961) extended the theory of special functions with matrix argument involving zonal polynomial. James (1961) and Constantine (1963) have given series of papers on multivariate analysis. Mathai (1978) introduced Laplace transform of binomial distribution function for positive definite matrix. Recently Sethi and Mishra (1995) discussed 'Laplace Transform of the binomial distribution function ${}_1F_0$ (α; – ;X) associated with zonal polynomial' of symmetric matrix. In this paper generalized hypergeometric distribution function associated with zonal polynomial has been used to define new formula by virtue of Laplace transform of matrix argument. It is very interesting that result earlier by the authors Sethi and Mishra (1995) and Mathai (1991) follows a special case.

All the matrices are assumed to $p \times p$ order symmetric positive definite. Unless otherwise stated. $A > 0$ means A is positive definite.

$R(.)$ means that real part of (.). $\int_0^I (.)\, dA$ means that (.) is integrated out over the $p \times p$ symmetric matrix A, such that $A > 0$ and $I - A > 0$ that in all the eigenvalues of A are between 0 and 1. The notation $\|(.)\|$ stands for the norms of (.). Here since the matrices are symmetric positive definite the largest eigenvalues can be taken as the norms. $|(.)|$ stands for determination of (.), *tr*(.) denotes the *trace* of (.).

2. ZONAL POLYNOMIALS. Let S be a positive definite symmetric $p \times p$ matrix and $\phi(S)$ a polynomial in the element of S. Then the transformation

$$\phi(S) \to \phi(L^{-1} SL'^{-1}) \quad L \in GL(P) \qquad \dots (2.1)$$

defines a representation of the real linear group $GL(p)$ in the vector space of all polynomials in S. The space V_k of homogeneous polynomial of degree K, in invariant under the given transformation and decomposes into the direct sum of irreducible subspace

$$V_k = \Sigma_k + V_{k,k}$$

where $K = (k_1, k_2, \dots, k_p)$; $k_1 \geq k_2 \geq \dots \geq k_p \geq 0$ sums over all partitions of k into not more than p parts.

These subspaces are generated by the zonal polynomial $Z(S)$, being invariant under the orthogonal group i.e.

$$Z_k\,(HSH') = Z_k(S)\,,\ H \in 0(p)\,.$$

They are homogeneous symmetric polynomials in the characteristic roots of S.

The fundamental property of the zonal polynomials is given by the following integral,

$$\int_{o(p)} C_k\,(H'TSH)\,dH = C_k(S)\,C_k(T)/C_k(I) \qquad \dots (2.2)$$

where I is the identity matrix of dH, is invariant Haar measure on the othogonal group.

Hypergeometric function of matrix will be defined as certain series of zonal polynomials. The equivalance of the two sets of functions follows from the important integral identity.

$$\int_{S>0} \exp\,(-\,tr\,RS)\,|S|^{-t-(p+1)/2}\,C_k\,(ST)\,dS$$

$$= \Gamma_p\,(t, k)\,C_k\,(R^{-1}\,T)\,|R|^{-t} \qquad \dots (2.3)$$

where $C_k(S)$ in the zonal polynomial corresponding to the partitions $k = (k_1, k_2, \dots, k_p)$; $k_1 \geq k_2 \geq \dots \geq k_p \geq 0$ of the integer k, t is the complex number satisfying $R(t) > \frac{(p-1)}{2}$ and $\Gamma_p(t, k) = \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma(t + k_i - \frac{(i-1)}{2})$, the integration is over the space of positive definite symmetric $p \times p$ matrices.

3. SOLUTION OF LAPLACE TRANSFORM OF THE DISTRIBUTION FUNCTION $_pF_q$

Theorem . If X, T and B are real positive definite symmetric matrix of order $p \times p$. If $k = (k_1, k_2, ..., k_p)$; $k_1 \geq k_2 \geq ... \geq k_p \geq 0$ runs over all partitions into not more than p parts, $C_k(S)$ the corresponding zonal polynomial. The Laplace transform of the generalized hypergeometric function $_pF_q$ is

$$L_f(T) = \frac{|T+B|^{-t} \, _pF_q \, [\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; (T+B)^{-1}]}{|B|^{-t} \, _pF_q \, [\alpha_1, ... \alpha_p; \beta_1, ..., \beta_q; (B^{-1})]} \quad ... (3.1)$$

here the distribution function $f(x)$ is given as

$$f(x) = \frac{|X|^{t-(p+1)/2} e^{-tr(BX)} \, _pF_q \, [\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; X]}{\Gamma_p(t, k) \, |B|^{-t} \, _pF_q \, [\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; B^{-1}]} \quad ... (3.2)$$

Proof : We consider the integral

$$\int_{x>0} |X|^{t - \frac{(p+1)}{2}} e^{-tr(BX)} \, _pF_q \, [\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; X] \, dX \quad ... (3.3)$$

where the generalized hypergeometric function $_pF_q$ in terms of zonal polynomial defined as

$$_pF_q \, [\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; X] = \sum_{k=0}^{\infty} \sum_{k} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \, k!} C_K(X) \quad ... (3.4)$$

here the zonal polynomial $C_K(X)$ is the component of $e^{tr(X)^k}$.

Then the equation (3.3) becomes

$$\int_{X>0} |X|^{t-(p+1)/2} e^{-tr(BX)} \, _pF_q \, [\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; X] \, dX$$

$$= \int_{X>0} |X|^{t-(p+1)/2} e^{-tr(BX)} \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \, k!} C_K(X) \, dX \quad ... (3.5)$$

$$= \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k\, k!} \int_{X>0} |X|^{t-(p+1)/2} e^{-tr(BX)} C_k(X) dX$$

By the virtue of the integral

$$\int_{X>0} |X|^{t-\frac{(p+1)}{2}} e^{-tr(BX)} C_K(X)\, dX = \Gamma_p(t, k)\, |B|^{-t} C_K(B^{-1})$$

the equation (3.5) becomes

$$= \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k\, k!} \Gamma_p(t, k)\, |B|^{-t} C_K(B^{-1})$$

$$= \Gamma_p(t, k)\, |B|^{-t} \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k\, K!} C_K(B^{-1})$$

$$= \Gamma_p(t, k)\, |B|^{-t}\, {}_pF_q\, [\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; B^{-1}] \qquad ... (3.6)$$

Now *p.d.f.* is defined as

$$f(x) = \frac{|X|^{t-(p+1)/2} e^{-tr(BX)}\, {}_pF_q\, [\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; X]}{\Gamma_p(t, k)\, |B|^{-t}\, {}_pF_q\, [\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; B^{-1}]}$$

for $X = X' > 0$, $B = B' > 0$, $\| X \| < 1$, $\| B \| < 1$ and $R(t) > (p-1)/2$

$= 0$, elsewhe e ... (3.7)

$$f(X) \geq 0\ \forall X \quad \int f(X)\, dX = 1$$

taking Laplace transfor n on both sides of equation (3.7), we have

$$\int_{X>0} e^{-tr(TX)} f(X)\, dX = \int_{X>0} e^{-tr(TX)} \frac{|X|^{t-(p+1)/2} e^{-tr(BX)}}{\Gamma_p(t, k)\, |B|^{-t}} \times \frac{{}_pF_q\, [\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q\, X]\, dX}{{}_pF_q\, (\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; B^{-1})}.$$

$$L_f(T) = \eta \int_{X>0} |X|^{t-(p+1)/2} e^{-tr(T+B)X}\, {}_pF_q\, (\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; X)\, dX$$

where $L_f(T) = \int_{x>0} e^{-tr(TX)} f(X)\, dX$

and $\eta = [\Gamma_p(t, k)\, |B|^{-t}\, {}_pF_q\, (\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; B^{-1}]^{-1}$

or $$L_f(T) = \eta \int_{X>0} |X|^{t-(p+1)/2}\, e^{-tr(T+B)X}$$

$$\sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \, k!} C_K(X) \, dX$$

$$L_f(T) = \eta \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \, k!} \int_{X>0} |X|^{t-(p+1)/2}$$

$$\times e^{-tr\,(T+B)\,X} C_K(X) \, dX \quad \ldots (3.8)$$

Putting

$(T+B)X = Y \qquad \Rightarrow tr(T+B)X = tr(Y)$

$X = (T+B)^{-1} Y \qquad \Rightarrow dX = |T+B|^{-(p+1)/2} dY$

$|X| = |T+B|^{-1} |Y| \Rightarrow |X|^{t-(p+1)/2} = |T+B|^{-t+(p+1)/2} |Y|^{t-(p+1)/2}$

when $X = 0 \qquad \Rightarrow Y = 0$.

Putting these values in equation (3.8) it becomes

$$L_f(T) = \eta \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \, k!} \int_{Y>0} |T+B|^{-t+(p+1)/2}$$

$$\times |Y|^{t-(p+1)/2} e^{-tr\,(Y)} C_K [(T+B)^{-1} Y] \; |T+B|^{-(p+1)/2} dY$$

or $$L_f(T) = \eta \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \, k!} \int_{Y>0} |Y|^{-t+(p+1)/2}$$

$$\times e^{-tr(Y)} C_K [(T+B)^{-1} Y] \, dY \quad \ldots (3.9)$$

Constantine [6] has proved a integral relating zonal polynomial with gamma function as

$$\int_{X>0} e^{-tr(RX)} |X|^{t-(p+1)/2} C_K (TX) \, dX = \Gamma_p (t,k) \, C_K(R^{-1} T) \; |R|^{-t}$$

This yields the result $R = I$ this equation gives

$$\int_{X>0} |X|^{t-(p+1)/2} e^{-tr(X)} \; C_K(XT) \, dX = \Gamma_p (t, k) \, C_K(T) \quad \ldots (3.10)$$

using the result (3.10) in equation (3.9), we get

$$L_f(T) = \eta \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k \, k!} \frac{|T+B|^{-t}}{k!} \Gamma_p (t, k) \, C_k [(T+B)^{-1}]$$

Putting the value of η we get

$$L_f(T) = \frac{|T+B|^{-t}\,\Gamma_p(t,k) \sum\limits_{k=0}^{\infty} \sum\limits_{K} \frac{(\alpha_1)_k \ldots (\alpha_p)_k\, C_K[(T+B)^{-1}]}{(\beta_1)_k \ldots (\beta_q)_k \quad k!}}{\Gamma_p(t,k)\,|B|^{-1}\,{}_pF_q\,[\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; B^{-1}]}$$

$$L_f(T) = \frac{|T+B|^{-t}\,{}_pF_q\,[\alpha_1, \ldots, \alpha_p\,; \beta_1, \ldots, \beta_q; (T+B)^{-1}]}{|B|^{-t}\,{}_pF_q\,[\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; B^{-1}]} \quad \ldots (3.11)$$

or $\quad L_f(T) = \left|\frac{T+B}{B}\right|^{-t} \chi$

where $\chi = \dfrac{{}_pF_q\,[\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; (T+B)^{-1}]}{{}_pF_q\,[\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; B^{-1}]}$

or $\quad L_f(T) = |I + B^{-1}\,T|^{-t}$

$\quad = |I + B^{-1/2}\,T\,B^{-1/2}|^{-t}$

The moment generating function of $f(X)$ is

$$|I + B^{-1}\,T|^{-t} \text{ and } I - B^{-1}\,T > 0.$$

4. SPECIAL CASES

(1) When we put $p = 0$, $q = 0$ in equation (3.11), the equation reduces to exponential series

$$L_f(T) = \frac{|T+B|^{-t}}{|B|^{-t}}\;\frac{{}_0F_0\,[(T+B)^{-1}]}{{}_0F_0\,[(B)^{-1}]}.$$

(2) When $p = 1$ and $q = 0$ equation (3.11) is reduced to the Laplace transform of binomial function ${}_1F_0\,(\alpha\,; -X)$ as

$$L_f(T) = \frac{|T+B|^{-t}}{|B|^{-t}}\;\frac{{}_1F_0\,[\alpha\,;-\,;(T+B)^{-1}]}{{}_1F_0\,[\alpha\,;-\,;(B^{-1})]}$$

(3) When $p = 1$ and $q = 1$, equation (3.11) is reduced to the Laplace transform of confluent hypergeometric function ${}_1F_1\,(\alpha; \gamma\,; X)$ as

$$L_f(T) = \frac{|T+B|^{-t}}{|B|^{-t}}\;\frac{{}_1F_1\,[\alpha; \gamma; (T+B)^{-1}]}{{}_1F_1\,[\alpha; \gamma; (B^{-1})]}.$$

(4) When $p = 2$ and $q = 1$ equation (3.11) reduces to the Laplace transform of Gaussian hypergeometric function ${}_2F_1\,(\alpha, \beta; \gamma; X)$ as

$$L_f(T) = \frac{|T+B|^{-t} \ {}_2F_1 [\alpha, \beta; \gamma; (T+B)^{-1}]}{|B|^{-t} \ {}_2F_1 [\alpha, \beta; \gamma; (B^{-1})]}$$

When $K = 0$ in all the investigations, we have probability density for gamma function due to Mathai and Saxena [4]

$$f(X) = \frac{|X|^{t-(p+1)/2} e^{-tr(BX)}}{\Gamma_p(t) \ |B|^{-t}}$$

where $X = X' > 0$ and $B = \beta' > 0$ and its Laplace transform is given as

$$L_f(T) = |I + B^{-1} T|^{-t} = |I + B^{-1/2} TB^{-1/2}|^{-t}$$

The moment generating function of $f(X)$ is

$$|I + B^{-1}T|^{-t}; I - B^{-1} T > 0.$$

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