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CERTAIN INTEGRALS INVOLVING HYPERGEOMETRIC FUNCTIONS OF THREE AND FOUR VARIABLES

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ABSTRACT

In the present paper some integrals involving hypergeometric functions of three and four variables are evaluated. Some particular cases are of interest.

1. INTRODUCTION Let

$$E_\alpha f(\alpha) = f(\alpha+1),\ E_\alpha^m f(\alpha) = f(\alpha+m) \qquad \text{...(1.1)}$$

Recently Agarwal [1] employed the operator E_α to evaluate the integrals

$$\int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\,\Gamma(\alpha+x)\,\Gamma(\beta-x)}\, {}_1F_1\left[\begin{matrix}\alpha'; \\ \alpha+x;\end{matrix}\ v\right] {}_1F_1\left[\begin{matrix}\beta'; \\ \beta-x;\end{matrix}\ v\right] dx;$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\, {}_1F_1\left[\begin{matrix}\alpha'+\beta'; \\ \alpha+\beta-1;\end{matrix}\ 2v\right] \qquad \text{...(1.2)}$$

$\mathrm{Re}(\alpha+\beta) > 1$, n is an integer.

$$\int_{-\infty}^{\infty} \frac{1}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)}\, {}_1F_1\left[\begin{matrix}\alpha+\beta-1; \\ \alpha+x;\end{matrix}\ v\right] {}_1F_1\left[\begin{matrix}\gamma+\delta-1; \\ \gamma+x;\end{matrix}\ v\right] dx$$

$$= \frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)}$$

$$ {}_2F_2\left[\begin{matrix}\frac{\alpha+\beta+\gamma+\delta-3}{2}, & \alpha+\beta+\gamma+\delta-3; \\ \beta+\gamma-1, & \delta+\alpha-1;\end{matrix}\; 4v\right] \qquad ...(1.3) $$

$\mathrm{Re}(\alpha+\beta+\gamma+\delta) > 3$.

Ragab and Simary [7] have also evaluated these integrals, by a different technique.

In the present paper, following integrals are evaluated, which involve hypergeometric functions of three and four variables :

$$ (I) \quad \int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\,\Gamma(\alpha+x)\Gamma(\beta-x)} H_B(\beta+\gamma, \gamma+\alpha, \alpha+\beta; \gamma, \alpha+x, \beta-x; u, v, w)\,dx $$

$$ = \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} H_A(\beta+\gamma, \gamma+\alpha, \alpha+\beta; \gamma, \alpha+\beta-1; u, 2v, 2w) $$

$$ \mathrm{Re}(\alpha+\beta) > 1, $$

$$ (II) \quad \int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\,\Gamma(\alpha+x)\Gamma(\beta-x)} F_K(\gamma', \alpha+\beta, \alpha+\beta, \gamma+\beta, \alpha', \gamma+\beta; \gamma, \alpha+x, \beta-x; u, v, w)\,dx $$

$$ = \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} F_M(\gamma', \alpha+\beta, \alpha+\beta, \gamma+\beta, \alpha', \gamma+\beta, \gamma, \alpha+\beta-1; u, 2v, 2w) $$

$$ \mathrm{Re}(\alpha+\beta) > 1. $$

$$ (III) \quad \int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\,\Gamma(\alpha+x)\Gamma(\beta-x)} F_A^{(3)}(\alpha', \gamma, \alpha+\beta, \gamma', \alpha+x, \beta-x; u, v, w)\,dx $$

$$ = \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} F_G(\alpha', \alpha', \alpha', \gamma, \alpha, \beta; \gamma', \alpha+\beta-1, \alpha+\beta-1; u, 2v, 2w) $$

$$ \mathrm{Re}(\alpha+\beta) > 1. $$

$$ (IV) \quad \int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\Gamma(\alpha+x)\Gamma(\beta-x)} F_{29}^{(4)}\left(\begin{matrix}\gamma+\alpha, \gamma+\alpha, \beta+\delta, \beta+\delta, \alpha+\delta, \gamma+\beta, \alpha+\delta, \gamma+\beta; \\ \gamma+\delta', \alpha+x, \beta-x, \gamma+\delta';\end{matrix}\; u, v, w, t\right) dx $$

$$ = \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} $$

$$ F_{58}^{(4)}\left(\begin{matrix}\gamma+\alpha, \gamma+\alpha, \beta+\delta, \beta+\delta, \alpha+\delta, \gamma+\beta, \alpha+\delta, \gamma+\beta; \\ \gamma+\delta', \alpha+\beta-1, \alpha+\beta-1, \gamma+\delta';\end{matrix}\; u, 2v, 2w, t\right) $$

$$ \mathrm{Re}(\alpha+\beta) > 1. $$

(V) $\int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\,\Gamma(\alpha+x)\Gamma(\beta-x)}$

$F_{28}^{(4)}\left(\begin{matrix}\gamma+\delta, \gamma+\delta, \alpha+\beta, \alpha+\beta, \gamma+\alpha, \delta+\beta, \gamma+\alpha, \delta+\beta;\\ \alpha+x, \beta-x, \gamma'+\delta', \gamma'+\delta';\end{matrix}\, u, v, w, t\right)dx$

$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}$

$.F_{47}^{(4)}\left(\begin{matrix}\gamma+\delta, \gamma+\delta, \alpha+\beta, \alpha+\beta, \gamma+\alpha, \delta+\beta, \gamma+\alpha, \delta+\beta;\\ \gamma'+\delta', \gamma'+\delta', \alpha+\beta-1, \alpha+\beta-1;\end{matrix}\, u, v, 2w, 2t\right)$

$\text{Re}(\alpha+\beta) > 1.$

(VI) $\int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\Gamma(\alpha+x)\Gamma(\beta-x)}$

$.F_{27}^{(4)}\left(\begin{matrix}\gamma+\alpha, \gamma+\alpha, \delta+\beta, \delta+\beta, \gamma'+\alpha', \gamma'+\alpha', \delta'+\beta', \delta'+\beta';\\ \gamma+\delta, \alpha+x, \gamma+\delta, \beta-x;\end{matrix}\, u, v, w, t\right)dx$

$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}$

$.F_{47}^{(4)}\left(\begin{matrix}\gamma+\alpha, \gamma+\alpha, \beta+\delta, \beta+\delta, \gamma+\alpha', \gamma'+\alpha', \delta'+\beta', \delta'+\beta';\\ \gamma+\delta, \alpha+\beta-1, \gamma+\delta, \alpha+\beta-1;\end{matrix}\, u, 2v, w, 2t\right)$

$\text{Re}(\alpha+\beta) > 1.$

where H_A, H_B, H_C are Srivastava's functions of three variables [cf. [9]; p.43], $F_G, F_K, F_M, F_A^{(3)}$ are Lauricella's functions of three variables [cf. [9]; p. 42, 33] and $F_{27}^{(4)}, F_{28}^{(4)}, F_{29}^{(4)}, F_{27}^{(4)}, F_{57}^{(4)}, F_{58}^{(4)}$ are Lauricella's functions of four variables [cf. [8]; p. 125-129].

In what follows, we will employ the following results [cf. [2]; p.224].

$$\int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\,\Gamma(\alpha+x)\Gamma(\beta-x)}\,dx = \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} \qquad \text{...(1.4)}$$

2. ANALYSIS OF THE RESULTS

To prove (*I*), multiply both sides of (1.4) by

$\frac{\Gamma(\gamma+\alpha)\Gamma(\alpha+\beta)\Gamma(\beta+\gamma)}{\Gamma(\gamma)}$ and apply the operator

$\exp(uE_\alpha E_{\alpha'} + vE_\beta E_{\beta'} + wE_\gamma E_{\gamma'})$; we obtain

$\int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]\,\Gamma(\gamma+\alpha)\Gamma(\beta+\gamma)}{\sin(\pi x)\,\Gamma(\alpha+x)\,\Gamma(\beta-x)\Gamma(\gamma)}$

$$\sum_{m,n,p=0}^{\infty} \frac{(\beta+\gamma)_{m+p}(\gamma+\alpha)_{m+n}(\alpha+\beta)_{n+p}}{(\gamma)_m\,(\alpha+x)_m\,(\beta-x)_n} \frac{u^m}{m!}\frac{v^n}{n!}\frac{w^p}{p!}\,dx$$

$$= \frac{2^{\alpha+\beta-2}\,\Gamma(\gamma+\alpha)\Gamma(\alpha+\beta)\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta-1)\Gamma(\gamma)}$$

$$\sum_{m,n,p=0}^{\infty} \frac{(\beta+\gamma)_{m+p}(\gamma+\alpha)_{m+n}(\alpha+\beta)_{n+p}}{(\gamma)_m\,(\alpha+\beta-1)_{m+n}} \frac{u^m}{m!}\frac{(2v)^n}{n!}\frac{(2w)^p}{p!}\,dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} H_A\,(\beta+\gamma,\gamma+\alpha,\alpha+\beta;\gamma,\alpha+\beta-1;u,2v,2w)$$

which completes the proof of (I).

Now consider (1.4), multiply both the sides by

$\frac{\Gamma(\gamma+\alpha)\Gamma(\beta+\delta)\Gamma(\gamma+\beta)\Gamma(\alpha+\delta)}{\Gamma(\gamma'+\delta')}$ and operate both the sides by

$\exp\,(uE_\gamma E_{\gamma'} + vE_\alpha E_{\alpha'} + wE_\beta E_{\beta'} + tE_\delta E_{\delta'})$, we get

$$\int_{-\infty}^{\infty} \frac{\sin\,[(2n+1)\pi x]\;\Gamma(\gamma+\alpha)\Gamma(\beta+\delta)\Gamma(\gamma+\beta)\Gamma(\alpha+\delta)}{\sin\,(\pi x)\;\Gamma(\alpha+x)\;\Gamma(\beta-x)\Gamma(\gamma'-\delta')}$$

$$\sum_{m,n,p,q=0}^{\infty} \frac{(\gamma+\alpha)_{m+n}(\beta+\delta)_{p+q}(\gamma+\beta)_{m+p}(\alpha+\delta)_{n+q}}{(\gamma'+\delta')_{m+q}\,(\alpha+x)_n\,(\beta-x)_p} \frac{u^m}{m!}\frac{v^n}{n!}\frac{w^p}{p!}\frac{t^q}{q!}\,dx$$

$$= \frac{2^{\alpha+\beta-2}\,\Gamma(\gamma+\alpha)\Gamma(\beta+\delta)\Gamma(\gamma+\beta)\Gamma(\alpha+\delta)}{\Gamma(\alpha+\beta-1)\Gamma(\gamma'+\delta')}$$

$$\sum_{m,n,p,q=0}^{\infty} \frac{(\gamma+\alpha)_{m+n}(\beta+\delta)_{p+q}(\gamma+\beta)_{m+p}(\alpha+\delta)_{n+q}}{(\gamma'+\delta')_{m+q}\,(\alpha+\beta-1)_{n+p}} \frac{u^m}{m!}\frac{(2v)^n}{n!}\frac{(2w)^p}{p!}\frac{t^q}{q!}$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}$$

$$F_{58}^{(4)}\left(\begin{matrix}\gamma+\alpha,\gamma+\alpha,\beta+\delta,\beta+\delta,\gamma+\beta,\gamma+\beta,\alpha+\delta,\alpha+\delta; \\ \gamma'+\delta',\alpha+\beta-1,\alpha+\beta-1,\gamma'+\delta';\end{matrix}\; u,v,2w,t\right)$$

which is the required result (IV).

By employing the operators

$$\exp\,(uE_\gamma E_{\gamma'} + vE_\alpha E_{\alpha'} + wE_\beta E_{\beta'}),$$

$$\exp\,(uE_\gamma E_{\gamma'} + vE_\alpha + wE_\beta E_{\beta'})E_{\alpha'}],$$

$$\exp\,(uE_\gamma E_{\gamma'} + vE_\delta E_{\delta'} + wE_\alpha E_{\alpha'} + tE_\beta E_{\beta'})$$

and $\exp\,(uE_\gamma E_{\gamma'} + vE_\alpha E_{\alpha'} + wE_\delta E_{\delta'} + E_\beta E_{\beta'})$ to (1.4).

we can easily obtain results (II), (III), (V) and (VI).

3. PARTICULAR CASES

By setting $u = 0$ in (I), following interested results are obtained

$$\int_{-\infty}^{\infty} \frac{[\sin (2n+1)\pi x]}{\sin (\pi x)\, \Gamma(\alpha+x)\Gamma(\beta-x)}$$

$$. F_2 (\alpha+\beta, \alpha+\beta, \gamma+\alpha, \beta+\gamma; \alpha+x, \beta-x; v, w)\, dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} F_1(\alpha+\beta, \alpha+\beta, \gamma+\alpha, \beta+\gamma; \alpha+\beta-1, \alpha+\beta-1; 2v, 2w) \quad ...(3.1)$$

For $w = 0$; (II) reduces to the following form

$$\int_{-\infty}^{\infty} \frac{[\sin (2n+1)\pi x]}{\sin (\pi x)\, \Gamma(\alpha+x)\Gamma(\beta-x)}$$

$$.F_3 (\gamma', \alpha+\beta, \gamma+\beta, \alpha'; \gamma, \alpha+x; u, v)\, dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} F_3(\gamma', \alpha+\beta, \gamma+\beta, \alpha'; \gamma, \alpha+\beta-1; u, 2v) \quad ...(3.2)$$

For $u = 0$; (II) reduces to the following interesting case

$$\int_{-\infty}^{\infty} \frac{[\sin (2n+1)\pi x]}{\sin (\pi x)\, \Gamma(\alpha+x)\Gamma(\beta-x)}$$

$$.F_2(\alpha+\beta, \alpha+\beta, \alpha', \gamma+\beta, \alpha+x; \beta-x; v, w)\, dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} F_2(\alpha+\beta, \alpha+\beta, \alpha', \gamma+\beta; \alpha+\beta-1, \alpha+\beta-1; 2v, 2w) \quad ...(3.3)$$

Finally, if we set $u = 0$ in (III), we obtain

$$\int_{-\infty}^{\infty} \frac{[\sin (2n+1)\pi x]}{\sin (\pi x)\, \Gamma(\alpha+x)\Gamma(\beta-x)}$$

$$. F_2 (\alpha', \alpha', \alpha, \beta; \alpha+x, \beta-x; v, w)\, dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} F_1(\alpha', \alpha', \alpha, \beta; \alpha+\beta-1, \alpha+\beta-1; 2v, 2w) \quad ...(3.4)$$

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