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ON SOME RELATIONS BETWEEN HYPERGEOMETRIC FUNCTIONS OF THREE AND FOUR VARIABLES

By

R.C. Singh Chandel and P.K. Vishwakarma

Department of Mathematics, D.V. Postgraduate College,
Orai - 285 001, U.P., India

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ABSTRACT

In the present paper, we shall establish certain relations involving hypergeometric functions of three variables introduced by Lauricella [4] (Also see Saran [5] and Srivastava and Karlsson [7, pp. 41-43]) and hypergeometric functions of four variables introduced by Exton [1,2,3] and Sharma-Parihar [6].

1. INTRODUCTION

Lauricella [4,p.114] introduced fourteen complete hypergeometric series in three variables of the second order. He denoted his triple hypergeometric series by the symbols $F_1, F_2, ..., F_{14}$ of which F_1, F_2, F_5 and F_9 correspond respectively to the three variables Lauricella series $F_A^{(3)}$, $F_B^{(3)}$, $F_C^{(3)}$ and $F_D^{(3)}$. After a gap of long time, Saran [5] initiated a systematic study of remaining ten series with the notations F_E, F_F, F_G, F_K, F_M, F_N, F_P, F_R, F_S and F_T for F_4, F_{14}, F_8, F_3, F_{11}, F_6, F_{12}, F_{10}, F_7 and F_{13} respectively.

Exton [1,2,3] introduced 21 complete hypergeometric series $K_1, K_2, ..., K_{21}$ of four variables. Sharma and Parihar [6] introduced 83 complete hypergeometric series $F_1, F_2, ..., F_{83}$ of four variables. It is remarkable that out of these 83 series, the following 19 series had already been appeared in the literature due to Exton [1,2,3] in the different notations :

$$F_9^{(4)} = K_1,\ F_1^{(4)} = K_2,\ F_{38}^{(4)} = K_3,\ F_{10}^{(4)} = K_4,\ F_2^{(4)} = K_5,\ F_{59}^{(4)} = K_6,\ F_{39}^{(4)} = K_7,$$
$$F_{11}^{(4)} = K_8,\ F_{12}^{(4)} = K_9,\ F_3^{(4)} = K_{10},\ F_{60}^{(4)} = K_{11},\ F_{40}^{(4)} = K_{12},\ F_{13}^{(4)} = K_{13},$$
$$F_{71}^{(4)} = K_{14}, F_{78}^{(4)} = K_{15}, F_{79}^{(4)} = K_{16}, F_{82}^{(4)} = K_{19}, F_{81}^{(4)} = K_{20}, F_{83}^{(4)} = K_{21}.$$

In the present paper, we shall establish certain relations involving above hypergeometric functions of three and four variables :

2. RELATIONS

Consider

$$(1-u)^{-b_1} F_A^{(3)}\left(a, b_1, b_2, b_3; c_1, c_2, c_3; \frac{x}{1-u}, y, z\right)$$

$$= \sum_{k=0}^{\infty} \frac{u^k}{k!} (b_1)_k F_A^{(3)} (a, b_1+k, b_2, b_3; c_1, c_2, c_3; x, y, z)$$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p} (b_1)_{m+q} (b_2)_n (b_3)_p (c_4)_q}{(c_1)_{m_1} (c_2)_n (c_3)_p (c_4)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^k}{k!}.$$

Therefore

(2.1) $$(1-u)^{-b_1} F_A^{(3)}\left(a, b_1, b_2, b_3; c_1, c_2, c_3; \frac{x}{1-u}, y, z\right)$$

$$= F_C^{(4)} (a, a, a, c_4, b_1, b_2, b_3, b_1; c_1, c_2, c_3, c_4; x, y, z, u),$$

where $|u| < 1, \left|\frac{x}{1-u}\right| + |y| + |z| < 1.$

Applying the same techniques and making slight adjustments in interchanging of variables, we derive the following relationships

(2.2) $$(1-u)^{-a} F_A^{(3)}\left(a, b_1, b_2, b_3; c_1, c_2, c_3; \frac{x}{1-u}, \frac{y}{1-u}, \frac{z}{1-u}\right)$$

$$= F_A^{(4)} (a, b_1, b_2, b_3, b_4; c_1, c_2, c_3, b_4; x, y, z, u)$$

$$|x| + |y| + |z| + |u| < 1.$$

(2.3) $$(1-u)^{-a_1} F_B^{(3)}\left(a_1, a_2, a_3, b_1, b_2, b_3; c; \frac{x}{1-u}, y, z\right)$$

$$= F_{76}^{(4)} (a_1, a_1, a_3, a_2, b_1, b_4, b_3, b_2; c, b_4, c, c; x, u, z, y)$$

$$|u| < 1, \left|\frac{x}{1-u}\right| < 1, |y| < 1, |z| < 1.$$

(2.4) $$(1-u)^{-a} F_C^{(3)}\left(a, b; c_1, c_2, c_3; \frac{x}{1-u}, \frac{y}{1-u}, \frac{z}{1-u}\right)$$

$$= K_2 (a, a, a, a; b, b, b, c_4; c_1, c_2, c_3, c_4; x, y, z, u)$$

$$|u| < 1, \left|\frac{x}{1-u}\right|^{1/2} + \left|\frac{y}{1-u}\right|^{1/2} + \left|\frac{z}{1-u}\right|^{1/2} < 1.$$

(2.5) $$(1-u)^{-a} F_D^{(3)}\left(a, b_1, b_2, b_3; c; \frac{x}{1-u}, \frac{y}{1-u}, \frac{z}{1-u}\right)$$

$$= K_{11}\,(a, a, a, a; b_1, b_2, b_3, b_4; c, c, c, b_4; x, y, z, u)$$

$$|u| < 1, \left|\frac{x}{1-u}\right| < 1, \left|\frac{y}{1-u}\right| < 1, \left|\frac{z}{1-u}\right| < 1.$$

(2.6) $$(1-u)^{-b_1} F_D^{(3)}\left(a, b_1, b_2, b_3; c; \frac{x}{1-u}, y, z\right)$$

$$= F_{64}^{(4)}\,(a, a, a, c', b_1, b_2, b_3, b_1; c, c, c, c'; x, y, z, u)$$

$$|u| < 1, \left|\frac{x}{1-u}\right| < 1, |y| < 1, |z| < 1.$$

(2.7) $$(1-u)^{-a_1} F_E\left(a_1, a_1, a_1, b_1, b_2, b_2; c_1, c_2, c_3; \frac{x}{1-u}, \frac{y}{1-u}, \frac{z}{1-u}\right)$$

$$= K_{10}\,(a_1, a_1, a_1, a_1; b_2, b_2, b_1, c_4; c_3, c_2, c_1, c_4; z, y, x, u),$$

$$|u| < 1, \text{ if } \left|\frac{x}{1-u}\right| < r, \left|\frac{y}{1-u}\right| < s, \left|\frac{z}{1-u}\right| < t$$

then $(1-s)(s-t) = rs$.

(2.8) $$(1-u)^{-b_2} F_E\left(a_1, a_1, a_1, b_1, b_2, b_2; c_1, c_2, c_2; x, \frac{y}{1-u}, \frac{z}{1-u}\right)$$

$$= F_4^{(4)}\,(a_1, a_1, a_1, c_4, b_2, b_2, b_1, b_2; c_3, c_2, c_1, c_4; z, x, y, u)$$

$$|u| < 1 \text{ if } |x| < r, \left|\frac{y}{1-u}\right| < s, \left|\frac{z}{1-u}\right| < t,$$

then $r + (\sqrt{s} + \sqrt{t})^2 = 1$.

(2.9) $$(1-u)^{-b_1} F_E\left(a_1, a_1, a_1, b_1, b_2, b_2; c_1, c_2, c_3; \frac{x}{1-u}, y, z\right)$$

$$= F_5^{(4)}\,(a_1, a_1, a_1, c_4, b_2, b_2, b_1, b_1; c_3, c_2, c_1, c_4; z, y, x, u)$$

$$|u| < 1, \text{ if } \left|\frac{x}{1-u}\right| < r, |y| < s, |z| < t,$$

then $r + (\sqrt{s} + \sqrt{t})^2 = 1$.

(2.10) $$(1-u)^{-a_1} F_F\left(a_1, a_1, a_1, b_1, b_2, b_1; c_1, c_2, c_2; \frac{x}{1-u}, \frac{y}{1-u}, \frac{z}{1-u}\right)$$

$$= K_8\,(a_1, a_1, a_1, a_1; b_1, b_1, b_2, c_3; c_2, c_1, c_2, c_3; y, x, z, u)$$

$$|u| < 1, \text{ if } \left|\frac{x}{1-u}\right| < r, \left|\frac{y}{1-u}\right| < s, \left|\frac{z}{1-u}\right| < t,$$

then $(1-s)(s-t) = rs$.

(2.11) $(1-u)^{-b_1} F_F\left(a_1, a_1, a_1, b_1, b_2, b_1; c_1, c_2, c_2; \frac{x}{1-u}, y, \frac{z}{1-u}\right)$

$= F_{15}^{(4)}(a_1, a_1, a_1, c_3, b_1, b_1, b_2, b_1; c_2, c_1, c_2, c_3; z, x, y, u)$

$|u| < 1$, if $\left|\frac{x}{1-u}\right| < r$, $|y| < s$, $\left|\frac{z}{1-u}\right| < t$,

then $(1-s)(s-t) = rs$.

(2.12) $(1-u)^{-b_2} F_F\left(a, a, a, b_1, b_2, b_1; c_1, c_2, c_2; x, \frac{y}{1-u}, z\right)$

$= F_{17}^{(4)}(a, a, a, c_3, b_1, b_1, b_2, b_2; c_2, c_1, c_2, c_3; z, x, y, u)$

$|u| < 1$, if $\left|\frac{y}{1-u}\right| < s$, $|x| < r$, $|z| < t$,

then $(1-s)(s-t) = rs$.

(2.13) $(1-u)^{-a} F_G\left(a, a, a\, b_1, b_2, b_3; c_1, c_2, c_2; \frac{x}{1-u}, \frac{y}{1-u}, \frac{z}{1-u}\right)$

$= K_{13}(a, a, a, a; b_3, b_2, b_1, c_3; c_2, c_2, c_1, c_3; z, y, x, u)$

$|u| < 1$, if $\left|\frac{x}{1-u}\right| < r$, $\left|\frac{y}{1-u}\right| < s$, $\left|\frac{z}{1-u}\right| < t$,

then $r+s=1, r+t=1$.

(2.14) $(1-u)^{-b_1} F_G\left(a, a, a, b_1, b_2, b_3; c_1, c_2, c_2; \frac{x}{1-u}, y, z\right)$

$= F_{23}^{(4)}(a, a, a, c_3, b_1, b_2, b_3, b_1; c_1, c_2, c_2, c_3; x, y, z, u)$

$|u| < 1$, if $|x| < r$, $\left|\frac{y}{1-u}\right| < s$, $|z| < t$

then $r+s=1, r+t=1$.

(2.15) $(1-u)^{-b_2} F_G\left(a, a, a, b_1, b_2, b_3; c_1, c_2, c_2; x, \frac{y}{1-u}, z\right)$

$= F_{22}^{(4)}(a, a, a, c_3, b_2, b_3, b_1, b_2; c_2, c_2, c_1, c_3; y, z, x, u)$

$|u| < 1$, if $|x| < r$, $\left|\frac{y}{1-u}\right| < s$, $|z| < t$,

then $r+s=1, r+t=1$.

(2.16) $(1-u)^{-a_1} F_K\left(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_3; \frac{x}{1-u}, y, z\right)$

$= F_8^{(4)}(a, a_1, a_2, a_2, b_1, c_4, b_1, b_2; c_1, c_4, c_3, c_2; x, u, z, y)$

$$|u| < 1 \text{ if } \left| \frac{x}{1-u} \right| < r,\ |y| < s,\ |z| < t$$

then $(1-r)(1-s) = t$.

(2.17) $(1-u)^{-a_2} F_K\left(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_3; x, \frac{y}{1-u}, \frac{z}{1-u} \right)$

$$|u| < 1,\ |x| < r,\ \left| \frac{y}{1-u} \right| < s,\ \left| \frac{z}{1-u} \right| < t$$

then $(1-s)(1-r) = t$.

(2.18) $(1-u)^{-a_1} F_M\left(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_2; \frac{x}{1-u}, y, z \right)$

$$= F_{31}^{(4)} (a_2, a_2, a_1, a_1, b_1, b_2, b_1, c_3; c_2, c_2, c_1, c_3; x, y, z, u)$$

$$|u| < 1, \text{ if } \left| \frac{x}{1-u} \right| < r,\ |y| < s,\ |z| < t$$

then $r + t = 1 - s$.

(2.19) $(1-u)^{-a_2} F_M\left(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_2; x, \frac{y}{1-u}, \frac{z}{1-u} \right)$

$$= F_{22}^{(4)} (a_2, a_2, a_2, a_1, b_1, b_2, c_3, b_1; c_2, c_2, c_3, c_1; z, y, x, u)$$

$$|u| < 1, \text{ if } |x| < r,\ \left| \frac{y}{1-u} \right| < s,\ \left| \frac{z}{1-u} \right| < t$$

then $F + t = 1 - s$.

(2.20) $(1-u)^{-b_1} F_M\left(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_2; \frac{x}{1-u}, y, \frac{z}{1-u} \right)$

$$= F_{25}^{(4)} (b_1, b_1, b_1, b_2, a_2, c_3, a_1, a_2; c_2, c_3, c_1, c_2; z, u, x, y)$$

$$|u| < 1 \text{ if } \left| \frac{x}{1-u} \right| < r,\ |y| < s,\ |z| < t$$

then $r + t = 1 - s$.

(2.21) $(1-u)^{-a_1} F_N\left(a_1, a_2, a_3, b_1, b_2, b_1; c_1, c_2, c_2; \frac{x}{1-u}, y, z \right)$

$$= F_{37}^{(4)} (b_1, b_1, c_3, a_2, a_1, a_3, a_1, b_2; c_1, c_2, c_3, c_2; x, z, u, y)$$

$$|u| < 1, \text{ if } \left| \frac{x}{1-u} \right| < r,\ |y| < s,\ |z| < t,$$

then $(1-r)s + (1-s)t = 0$.

(2.22) $(1-u)^{-a_2} F_N\left(a_1, a_2, a_3, b_1, b_2, b_1; c_1, c_2, c_2; x, \frac{y}{1-u}, z \right)$

$$= F_{35}^{(4)}(a_2, a_2, b_1, b_1, b_2, c_3, a_3, a_1; c_2, c_3, c_2, c_1; y, u, z, x)$$

$$|u| < 1, \text{ if } |x| < r, \left| \frac{y}{1-u} \right| < s, \ |z| < t,$$

then $(1-r)s + (1-s)t = 0.$

$$(2.23)\ (1-u)^{-b_1} F_N\left(a_1, a_2, a_3, b_1, b_2, b_1; c_1, c_2, c_2; \frac{x}{1-u}, y, \frac{z}{1-u} \right)$$

$$= F_{26}^{(4)}(b_1, b_1, b_1, b_2, a_3, c_3, a_1, a_2; c_2, c_3, c_1, c_2; z, u, x, y)$$

$$|u| < 1, \text{ if } \left| \frac{x}{1-u} \right| < r, \ |y| < s, \left| \frac{z}{1-u} \right| < t,$$

then $(1-r)s + (1-s)t = 0.$

$$(2.24)\ (1-u)^{-a_1} F_P\left(a_1, a_2, a_1, b_1, b_1, b_2; c_1, c_2, c_3, c_2; \frac{x}{1-u}, y, \frac{z}{1-u} \right)$$

$$= F_{24}^{(4)}(a_1, a_1, a_1, a_2, b_1, b_2, c_3, b_1; c_1, c_2, c_3, c_2; x, z, u, y)$$

$$|u| < 1, \text{ if } \left| \frac{x}{1-u} \right| < r, \left| \frac{z}{1-u} \right| < t, \ |y| < s$$

then $(st - s - t)^2 = 4\,rst.$

$$(2.25)\ (1-u)^{-a_2} F_P\left(a_1, a_2, a_1, b_1, b_1, b_2; c_1, c_2, c_2; x, \frac{y}{1-u}, z \right)$$

$$= F_{32}^{(4)}(a_2, a_2, a_1, a_1, b_1, c_3, b_1, b_2; c_2, c_3, c_1, c_2; y, u, x, z)$$

$$|u| < 1, \text{ if } |x| < r, \left| \frac{y}{1-u} \right| < s, \ |z| < t,$$

then $(st - s - t)^2 = 4\,rst.$

$$(2.26)\ (1-u)^{-b_1} F_P\left(a_1, a_2, a_1, b_1, b_1, b_2; c_1, c_2, c_3; \frac{x}{1-u}, \frac{y}{1-u}, z \right)$$

$$= F_{24}^{(4)}(b_1, b_1, b_1, b_2, a_1, a_2, c_3, a_1; c_1, c_2, c_3, c_2; x, y, u, z)$$

$$|u| < 1, \text{ if } \left| \frac{x}{1-u} \right| < r, \left| \frac{y}{1-u} \right| < s, \ |z| < t$$

then $(st - s - t)^2 = 4\,rst.$

(2.27) $(1-u)^{-a_1} F_R\left(a_1, a_2, a_1, b_1, b_2, b_1; c_1, c_2, c_2; \frac{x}{1-u}, y, \frac{z}{1-u}\right)$

$= F_{20}^{(4)}(a_1, a_1, a_1, a_2, b_1, b_1, c_3, b_2; c_1, c_2, c_3, c_2; x, z, u, y)$

$|u| < 1$, if $\left|\frac{x}{1-u}\right| < r$, $\left|\frac{z}{1-u}\right| < t$, $|y| < s$,

then $s(1-\sqrt{r})^2 + t(1-s) = 0$.

(2.28) $(1-u)^{-a_2} F_R\left(a_1, a_2, a_1, b_1, b_2, b_1; c_1, c_2, c_2; x, \frac{y}{1-u}, z\right)$

$= F_{30}^{(4)}(a_1, a_1, a_2, a_2, b_1, b_1, b_2, b_3; c_2, c_1, c_2, c_3; z, x, y, u)$

$|u| < 1$, if $|x| < r$, $\left|\frac{y}{1-u}\right| < s$, $|z| < t$,

then $r + s = rs$, $sr = t$.

(2.29) $(1-u)^{-a_1} F_S\left(\{a_1, a_2, a_2, b_1, b_2, b_3; c_1, \frac{x}{1-u}, y, z\right)$

$= F_{72}^{(4)}(a_2, a_2, a_1, b_3, b_2, b_1, c_2; c_1, c_1, c_1, c_2; z, y, x, u)$,

$|u| < 1$, if $\left|\frac{x}{1-u}\right| < r$, $|y| < s$, $|z| < t$,

then $r + s = rs$, $s = t$.

(2.30) $(1-u)^{-a_2} F_S\left(a_1, a_2, a_2, b_1, b_2, b_3; c_1, c_1, c_1; x, \frac{y}{1-u}, \frac{z}{1-u}\right)$

$= F_{67}^{(4)}(a_2, a_2; a_2, a_1, b_3, b_2, c_2, b_1; c_1, c_1, c_2, c_1; x, y, z, u)$

$|u| < 1$, if $|x| < r$, $\left|\frac{y}{1-u}\right| < s$, $\left|\frac{z}{1-u}\right| < t$,

then $r + s = rs$, $s = t$.

(2.31) $(1-u)^{-b_2} F_S\left(a_1, a_2, a_2, b_1, b_2, b_3; c_1, c_1, c_1; x, \frac{y}{1-u}, z\right)$

$= F_{74}^{(4)}(a_2, a_2, c_2, a_1, b_2, b_3, b_2, b_1; c_1, c_1, c_2, c_1, y, z, u, x)$

$|u| < 1$, if $|x| < r$, $\left|\frac{y}{1-u}\right| < s$, $|z| < t$,

then $r + s = rs$, $s = t$.

(2.32) $(1-u)^{-a_1} F_T\left(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_1, c_1; \frac{x}{1-u}, y, z\right)$

$= F_{70}^{(4)}(a_2, a_2, a_1, a_1, b_1, b_2, b_1, c_2; c_1, c_1, c_2; z, y, x, u)$

$$|u| < 1, \text{ if } \left| \frac{x}{1-u} \right| < r, \ |y| < t, \ |z| < t,$$

then $r + s = sr + t$.

$$(2.33)\ (1-u)^{-a_2} F_T\left(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_1, c_1; x, \frac{y}{1-u}, \frac{z}{1-u} \right)$$

$$= F_{65}^{(4)}\ (a_2, a_2, a_2, a_1, b_1, b_2, c_2, b_1; c_1, c_1, c_2, c_1; z, y, u, x)$$

$$|u| < 1, \text{ if } |x| < r, \ \left| \frac{y}{1-u} \right| < s, \ \left| \frac{z}{1-u} \right| < t,$$

then $r + s = rs + t$.

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