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GENERALIZED CONVOLUTION FOR *K*-TRANSFORMATION

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ABSTRACT

In this paper we extend the K-transformation to a class of generalized functions. We discuss the K-transformation as a special case of convolution transform and prove an inversion for it in distributional sense.

1. INTRODUCTION

The convenctional K-transformation is defined by

$$F(s) = \int_0^{\infty} \sqrt{(st)}\, K_\mu\, (st)\, f(t)\, dt \qquad ...(1.1)$$

where $0 < t < \infty$, $0 < s < \infty$, $0 \leq \text{Re}\, \mu < 0$ and $K_\mu\, (z)$ denotes the modified Bessel function of third kind of order μ (Jahnke, Emde and Losch [6,p.207]). C.S. Meijer [8] was apprently the first to investigate the K-transformation. Other early investigations of (1.1) were made by Boas [1,2], Erdelyi [3]. In 1968, Zemanian [12] extended the conventional convolution transformation

$$F(x) = \int_{-\infty}^{\infty} f(t)\, G(x-t)\, dt \qquad ...(1.2)$$

to a class of generalized functions.

We define the K-transformation of a generalized function $f(t)$ by

$$F(s) = \langle f(t), \sqrt{(st)}\, K_\mu\, (st)\rangle \qquad ...(1.3)$$

In this paper we extend the transform (1.3) to a class of generalized functions as a special case of convolution transform.

2. THE TESTING FUNCTION SPACES D, $D(I)$, $L_{a,b}$ AND $M_{c,d}$

A function is said to be smooth if its derivatives of all order are continuous at all points of its domain. The space D consists of all complex-valued smooth functions $\phi(t)$ that are zero outside some finite interval. Let I be the inveral $0 < t < \infty$. $D(I)$ is the space of infinitely differentiable functions having compact support defined over the open interval I. S denotes the space of smooth functions of rapid descent.

Let $\eta_{a,b}(t)$ be the function

$$\eta_{a,b}(t) = \begin{matrix} e^{at} & 0 \le t < \infty \\ e^{bt} & -\infty < t < 0 \end{matrix} \qquad \text{...(2.1)}$$

$L_{a,b}$ denotes the space of all complex-valued smooth functions $\phi(t)$ on $-\infty < t < \infty$, on which the functionals γ_k defined by

$$\gamma_k \overset{\Delta}{=} \gamma_{a,b,k} \overset{\Delta}{=} \sup_{-\infty < t < \infty} |\eta_{a,b}(t)\, D^k\, \phi(t)| \qquad \text{...(2.2)}$$

($k = 0, 1, 2, \ldots$) assume finite values.

We assign to $L_{a,b}$ the topology generated by $\{\gamma_k\}_{k=0}^{\infty}$ thereby making it countably multinormed space.

Applying the change of variable $T = e^{-t}$, $t = -\log T$, $D_t = -TD_T$ to the definition of $L_{a,b}$ and setting $T\,\psi(T) = \phi(-\log T)$ in (2.1) and (2.2), we have the following defintion.

Given any two real numebrs c and d, $M_{c,d}$ is the space of all smooth functions $\psi(T)$ on $0 < T < \infty$ such that

$$i_k\, \psi(T) = i_{c,d,k}\, \psi(T)$$

$$= \sup_{0 < T < \infty} |\eta_{c,d}(-\log T)(-D_T)^k\{T\psi(T)\}| < \infty \quad (k = 0, 1, 2, \ldots)$$

where $$\eta_{c,d}(-\log T) = \begin{matrix} T^{-c} & 0 < T \le 1 \\ T^{-d} & 1 < T < \infty \end{matrix}$$

The topology of $M_{c,d}$ is that generated by the multinorm $\{i_k\}_{k=0}^{\infty}$. As a consequence, $M_{c,d}$ is a complete countably multinormed space. $D'(I)$ is the space of distributions defined over the testing function space D. The spaces $D'(I)$, $L'_{a,b}$ and $M'_{c,d}$ are topological duals of the spaces $D(I)$, $L_{a,b}$ and $M_{c,d}$ respectively.

Theorem 2.1 *$\sqrt{(st)}\, K_\mu(sT)$ is in $M_{c,d}$ for $c < 1$ and for every real d.*

Proof. For $\sqrt{(sT)}\, K_\mu(sT)$ to be in $M_{c,d}$, we have to show that

$$\sup_{0<T<\infty} |\eta_{c,d}(-\log T)(-TD_T)^k \{T\psi(T)\}|$$

is bounded, where $\psi(T) = \sqrt{(sT)}\, K_\mu(sT)$.

From Erdelyi [4,p.79], we have

$$Dz^{-\mu} K_\mu(z) = z^{-\mu} K_{\mu+1}(z).$$

Again from Erdelyi [4,p.24], we have

$$\sqrt{z}K_\mu(z) = \sqrt{\pi/2}e^{-z}\,[1 + O(|z|^{-1})] \text{ as } z \to \infty.$$

Hence

$$\sup_{0<T<\infty} |\eta_{c,d}(-\log T)(-TD_T)^k \{T\psi(T)\}|$$

$$= \sup_{0<T<\infty} |\eta_{c,d}(-\log T)(-TD_T)^k \{\sqrt{s}T^{\mu+3/2}\, T^{-\mu} K_\mu(sT)\}|$$

$$= \sup_{0<T<\infty} |\eta_{c,d}(-\log T)\sqrt{s}(-1)^k \sum_p Q_\mu T^{p+\mu+3/2} D_T^p \{T^{-\mu} K_\mu(sT)\}|$$

(Q_μ is a polynomial in μ, $0 \le p \le k$)

$$= \sup_{0<T<\infty} |\eta_{c,d}(-\log T)\sqrt{s}(-1)^k \sum_p Q_\mu (-1)^p T^{p+3/2} s^p K_{\mu+p}(sT)|$$

For $0 < T < 1$,

$$\sup_{0<T<1} |\eta_{c,d}(-\log T)(-TD_T)^k \{T\psi(T)\}|$$

$$= \sup_{1<T<\infty} |\sum_p (-1)^{k+p} T^{-c+1} Q_\mu (sT)^p \sqrt{(sT)}\, K_{\mu+p}(sT)| \to 0$$

as $T \to 0$, s fixed and $c < 1$ (Zemanian [12], p.172)

Now for $|T| \to \infty$

$$\sup_{1<T<\infty} |\eta_{c,d}(-\log T)(-TD_T)^k \{T\psi(T)\}|$$

$$= \sup_{1<T<\infty} |\sum_p (-1)^{k+p} T^{-d+1} Q_\mu (sT)^p \sqrt{(\pi/2)}\, e^{-sT}|$$

= a finite quantity for any value of d.

Thus $\sqrt{(st)}\, K_\mu(sT)$ is a member of $M_{c,d}$ for every $c < 1$ and every d.

Theorem 2.2 *The mapping*

$$\psi(T) \to e^{-t}\psi(e^{-t}) = \phi(t) \qquad \text{...(2.3)}$$

is an isomorphism from $M_{c,d}$ onto $L_{c,d}$. The inverse mapping is given by

$$\phi(t) \to T^{-1} \phi(-\log T) = \psi(T) \qquad ...(2.4)$$

Proof. The mappings (2.3) and (2.4) are linear and inverses of one another are obvious.

Let $\psi(T) \in M_{c,d}$. A simple computation shows that $D_t^k \{e^{-t} \psi(e^{-t} \psi(e^{-t})\}$ is equal to a finite sum of terms, a typical term being $a_p T^{p+1} D_T^p \psi(T)$, where $0 \le p < k$ and a_p is a constant. Thus

$$\eta_{c,d}(t) D_t^k \{e^{-t} \psi(e^{-t})\}$$

$$= \sum_p a_p \eta_{c,d} (-\log T) T^{p+1} D_T^p \psi(T)$$

$$= \sum b_p , \eta_{c,d} (-\log T) (-TD_{T)}^p \{T\psi(T)\} \qquad ...(2.5)$$

(where b_p is another constant) so that

$$\gamma_{c,d,k}^{(\phi)} = \gamma_{c,d,k} \{e^{-t} \psi (e^{-t})\}$$

$$\le \sum_p |b_p| \; i_{c,d,k} \{\psi(T)\} \qquad ...(2.6)$$

consequently, (2.3) is a continuous mapping of $M_{c,d}$ into $L_{c,d}$.

Now let $\phi(t) \in L_{c,d}$, a straight forward computation shows that

$$(-TD_T)^k [T^{-1} \phi(-\log T)] \le \sum_p |c_p| \; D_t^p \phi(t)$$

where $0 \le p \le k$ and c_p's are constants.

Therefore

$$i_{c,d,k} \{\psi(T)\} = i_{c,d,k} [T^{-1} \phi(-\log T)]$$

$$\le \sum_p |c_p| \; \gamma_{c,d,p} \{\phi\}$$

Thus (2.4) is a continuous linear mapping of $L_{c,d}$ into $M_{c,d}$.

Since the mappings (2.3) and (2.4) are one-one, we can now conclude that they are also onto. Our proof is complete.

The dual space $L'_{c,d}$ denotes the spae of continuous linear functions on $M_{c,d}$ by

$$< f(-\log T), T^{-1} \phi(-\log T) > = < f(t), \phi(t) > \phi(t) \in L_{c,d} \qquad ...(2.7)$$

Theorem 2.3 *Let ψ and ϕ be related by (2.3) and (2.4). The mapping $f(t) \to f(-\log T)$, which is defined by (2.7), is an isomorphism from $L_{c,d}$ to $M_{c,d}$. The inverse mapping $f(T) \to f(e^{-t})$ is defined by*

$$<f(e^{-t}), \phi(t)> = <f(T), \psi(T)> \qquad ...(2.8)$$

Proof. The proof is parallel to that of Theorem 4.2-2 [Zemanian, 12, p. 105].

3. THE DISTRIBUTIONAL *K*-TRANSFORM

Let $G(t) = e^{t/2}\, e^t\, K_\mu\, (e^t)$. Setting $y = e^x$, $T = e^{-t}$. and $\phi(t) = G(x - t)$, we obtain

$$T^{-1}\, \phi\, (-\log T) = e^t\, G(x-t) = \sqrt{(yT)}\, y\, K_\mu\, (yT).$$

If we choose $c < 1$ and d any real number, we may replace $\phi(t)$ by $G(x - t)$ in (2.7) to obtain

$$< f(-\log T), y\sqrt{(yT)}K_\mu(yT) > = < f(t), G(x-t) >= F(\log y).$$

We finally obtain the new definition of the distributional *K*-transform as

$$J(y) = <j(T), \sqrt{(yT)}K_\mu(yT) >, 0 < y < \infty. \qquad ...(3.1)$$

This has a meaning as an application of $j(T) \in M_{c,d}$ to $\sqrt{(yT)}\; K_m\; (yT) \in M_{c,d}$, where $c < 1$ and d is arbitrary.

Theorem 3.1 *If*

$$F(s) = <f(T), \sqrt{(sT)}\; K_\mu\; (sT) > \qquad ...(3.2)$$

then $F(s)$ *is smooth function on* $0 < s < \infty$ *and*

$$F^k(s) = < f(T), \frac{\partial^k}{\partial s^k} \sqrt{(sT}K_\mu\, (sT) > \qquad ...(3.3)$$

Proof. We have

$$\frac{\partial^k}{\partial s^k} \{\sqrt{(sT)}K_\mu(sT)\} = \sum_{r=0}^{k} (k)_r\, (-1)^r\, (\mu + \tfrac{1}{2})_{k-r} (sT)^{r + 1/2} s^{-k} K_{\mu + r}(sT).$$

It can be seen that $\frac{\partial k}{\partial s^k} \sqrt{(sT)}K_\mu(sT) \in M_{c,d}$. Hence $M_{c,d}$ is closed under differenciation so that right hand side of (3.3) has sense. Thus, we need merely prove (3.3) is true for k replaced by $k - 1$. It is true definition for $k = 0$. Letting s fixed and $s \neq 0$, let us consider

$$\frac{F^{k-1}(s + \Delta s) - F^{k-1}(s)}{\Delta s} - < f(T), \{\sqrt{(sT)}K_\mu(sT)\}>=<f(T), \psi_{\Delta s}(T)> \qquad ...(3.4)$$

where $\psi_{\Delta s}\,(T) = \frac{1}{\Delta s}\, [\frac{\partial^{k-1}}{\partial s^{k-1}} \{\sqrt{(s} + \Delta s)TK_\mu\overline{(s + \Delta sT)}\}$

$$-\frac{\partial^{k-1}}{\partial s^{k-1}}\{\sqrt{(sT)}K_\mu(sT)\}] - \frac{\partial^k}{\partial s^k}\sqrt{(sT)}K_\mu(sT).$$

Proceeding as in Zemanian [12,p.235], it can be seen that $\psi_{\Delta s}(T)$ converges to zero as $\Delta s \to 0$. Therefore (3.4) converges to zero as $\Delta s \to 0$. This completes the proof of (3.2).

4. INVERSION FORMULA FOR THE DISTRIBUTIONAL *K*-TRANSFORM

In this section, we find out an inversion formula for the *K*-transform (1.1) by putting it into the form of convolution transform. The technique employed in finding the inversion formula is as given by Zemanian [12 p.229-246]. The same technique is also employed by O.P. Mishra [9] and A.K. Tewari [10].

Zemanian [12, pp. 229-246] has extended the convenctional convolution transform (1.2) to a class of generalized functions. He has shown that its inversion formula is still valid when the limiting operation in that formula is understood as weak convergence in the space D' of Schwartz distributions.

The conventional *K*-transform is

$$F(s) = \int_0^\infty \sqrt{(sT)}\, K_\mu\,(sT)\, f(T)\, dT \qquad ...(4.1)$$

or

$$e^s F(e^s) = \int_{-\infty}^\infty e^{3/2(s-t)}\, K_\mu\,(e^{s-t})\, f(e^{-t})\, dt$$

or

$$\tilde{p}(s) = \int_{-\infty}^\infty e^{3/2(s-t)}\, K_\mu(e^{s-t})\, P(t)\, dt \qquad ...(4.2)$$

where $\tilde{p}(s) = e^s F(e^s)$ and $p(t) = f(e^{-t})$.

Proceeding as in Joshi [7] and using some results from Hirschman and Widder [5, p. 66], the inversion operator is given by

$$E(D)[e^s F(e^s)] = E(D)[\tilde{p}(s)]$$

$$= [\Gamma(n+1)]^2 2^{2n} (2n)^{-1/2-s} e^{(\mu+½)s} D_1^{-n} e^{(-n-2\mu)s}$$

$$D_1^{-n} e^{(\mu-n+½)s} F(e^s)$$

$$= f(e^s)$$

where $D_1 = \dfrac{d}{de^s}.$

Returning to the original variables, the inversion operator for (4.1) is given by

$$\lim_{n \to \infty} [\Gamma(n+1)]^2 \, 2^{2n} \, (2n)^{-1/2} - \log s \; s^{\mu + 1/2} D^{-n} s^{-n-2\mu} D^{-n} s^{\mu - n + 1/2} F(s)$$

$$= f(s) \qquad \ldots(4.3)$$

This has a sense as a limit in $D(I)$. The change of variable we have used in (2.7), defines as isomorphism from D' onto $D'(I)$. In summary, if $p \in M'_{c,d}$ and if $\tilde{p}$ is defined by (4.2) then (4.3) holds true in sense of convergence in $D'(I)$.

As a consequnece of the inversion formula, we have

Theorem 4.1 (The Uniqueness Theorem) : Let $f \in L'_{c,d}$ and $h \in L'_{c,d}$. Also let $F(s) = \langle f(t), G(x-t) \rangle$ and $H(x) = \langle h(t), G(x-t) \rangle$. If $F(x)$ for all x, then $f = h$ in the sense of equality in D'.

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