Jñānābha, Vol.26,1996

BOUNDARY LAYER FLOW OF WALTERS LIQUID *B* THROUGH POROUS MEDIA

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(*Received : May 10, 1993*)

ABSTRACT

A boundary layer flow of Walters liquid *B* model is considered through porous medium. The governing equations of flow have been solved by *L-T* method for velocity field when pressure gradient is constant, varying linearly and harmonically. The influence of visco-elasticity is studied on velocity field when pressure gradient is constant. Also, the effect of permeability of medium is discussed graphically on velocity field in the same case.

1. INTRODUCTION

The flow of viscous fluids through porous media is of considerable interest in the fields of petrolium engineering, ground water hydrology and heat transfer in cooling systems. The analysis and the solution of the most of the problems of porous media flow depend upon Darcy's law. The validity of Darcy's law is subject to several limitations. A better understanding of these limitations is discussed by Whitakar in 1966. In 1967, Slattery studied the motion of a visco-elastic fluid through porous media. Ahmadi and Manvi [1] (1971) continued the study of viscous flow through a rigid porous medium and obtained the solution for some simple types of porous medium and obtained the solution for some simple types of flows. Gulab and Mishra [2] in 1977 worked for unsteady flow through *MHD* porous media and discussed the related resutls graphically. Harinath and Ramesan [3] in 1978 made an attempt to extend the work of Ahmadi and Manavi; and Gulab and Mishra to Rivilin-Ericksen fluid. In 1984, Siddappa and Abel [4] obtained velocity profile for the flow of Maxwellian fluid and the effect of time-relaxation parameter was discussed graphically.

This paper deals with the study of boundary layer flow of Walters liquid *B* through porous stratum applying *L-T* method and the influence of visco-elasticity and porousity of media have been discussed.

2. FORMATION OF THE PROBLEM

Let x-axis be parallel to the walls and y-axis be perpendicular to the walls. The walls are situated in the plane $y = \pm b$, where the walls are apart from each other by the distance of $2b$. Let u be the velocity of fluid in x-direction and be the function of y and t only due to infinite length of walls. The equation governing the flow of Walters liquid B be

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \upsilon \frac{\partial^2 u}{\partial y^2} - k_0^* \frac{\partial^3 u}{\partial t\, \partial y^2} - \frac{\upsilon}{k} u \qquad ...(2.1)$$

where k is permeability of medium and υ is kinematic viscosity.

The boundary conditions are

$$\begin{aligned} u &= 0 \quad \text{for} \quad t \le 0 \\ u &= 0 \quad \text{when } y = \pm b,\ t > 0 \\ u_y &= 0 \quad \text{when } y = 0 \end{aligned} \qquad ...(2.2)$$

3. SOLUTION OF THE PROBLEM

Introducing the dimensionless parameters given by

$$y' = y/b,\ t' = \upsilon t/b^2,\ u' = ub/\upsilon,\ x' = x/b$$

$$p' = pb^2/\rho\upsilon^2,\ k_0^{*'} = k_0^*/b^2 \text{ and } k' = k/b^2$$

Substituting all these in (2.1) and (2.2) and droping the dash for simplicity, we get

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} - k_0^* \frac{\partial^3 u}{\partial t\, \partial y^2} - \frac{u}{k} \qquad ...(3.1)$$

and boundary conditions

$$u = 0 \text{ for } t \le 0,\ y = \pm 1 \qquad ...(3.2)$$

$$u_y = 0,\ y = 0.$$

We applying L-T to solve the following three case as denoted by

$$\bar{u}\,(y, s) = \int_0^\infty e^{-st}\, u(y, t)\, dt \qquad ...(3.3)$$

Case I We assume that pressure gradient is constant i.e. $-\frac{\partial p}{\partial x} = A$ (positive). The equation (3.1) reduces to

$$\frac{\partial u}{\partial t} = A + \frac{\partial^2 u}{\partial y^2} - k_0^* \frac{\partial^3 u}{\partial t\, \partial y^2} - \frac{u}{k} \qquad ...(3.4)$$

Applying *L-T* to the equation (3.4), we have

$$s\bar{u} = \frac{A}{s} + \frac{d^2\bar{u}}{dy^2} - k_0^* s \frac{d^2\bar{u}}{dy^2} - \frac{\bar{u}}{k}$$

or

$$\frac{d^2\bar{u}}{dy^2} - \frac{s + 1/k}{1 - k_0^* s}\bar{u} = -\frac{A}{s\,(1 - k_0^* s)} \qquad ...(3.5)$$

whose solution subject to the boundary conditions (2.2) is

$$\bar{u}\,(y, s) = \frac{A}{s\,(s + 1/k)}\left[1 - \frac{\cosh\left(\dfrac{s + 1/k}{1 - k_0^* s}\right)^{1/2} y}{\cosh\left(\dfrac{s + 1/k}{1 - k_0^* s}\right)^{1/2} y}\right] \qquad ...(3.6)$$

Apply the inversion rule

$$u(y, t) = \frac{1}{2\pi i}\int_{a - i\infty}^{a + i\infty} e^{st}\,\bar{u}\,(y, s)\, ds$$

$$= \text{sum of the residues of } e^{st}\,\bar{u}\,(y, s)$$

$$= Ak\left[1 - \frac{\cosh\sqrt{1/k}\, y}{\cosh\sqrt{1/k}}\right]$$

$$+ \frac{4A}{\pi}\sum_{n=1}^{\infty}(-1)^n \frac{e^{s}\, n^{t} \cos{(2n+1)\,\pi/2y}}{(2n+1)\,[(2n+1)^2\,\pi^2/4 + 1/k]} \qquad ...(3.7)$$

Case II Pressure gradient varies linearly i.e.

$$-\frac{\partial P}{\partial x} = (A_0 + B_0 t).$$

The equation (3.1) becomes

$$\frac{\partial u}{\partial t} = (A_0 + B_0 t) + \frac{\partial^2 u}{\partial y^2} - k_0^* \frac{\partial^3 u}{\partial t\, \partial y^2} - \frac{u}{k} \qquad ...(3.8)$$

Laplace Transform of the equation (3.8) is given by

$$\frac{\partial^2\,\bar{u}\,(y, s)}{dy^2} - \left(\frac{s + 1/k}{1 - k_0^* s}\right)\bar{u}\,(y, s) = -\frac{A_0 s - B_0}{s^2\,(1 - k_0^* s)} \qquad ...(3.9)$$

whose solution subject to the boundary conditions (3.2) is

$$\bar{u}(y, s) = \frac{A_0 s + B_0}{s^2 (s + 1/k)} \left[1 - \frac{\cosh \left(\frac{s + 1/k}{1 - k_0^* s} \right)^{1/2}}{\cosh \sqrt{1/k}} \right] \quad ...(3.10)$$

Applying the inversion rule, we get

$$u(y, t) = \frac{\begin{array}{l}(A_0/k)(\cosh^2 \sqrt{1/k} - \cosh \sqrt{1/k}y \cosh \sqrt{1/k} \\ + B_0\{t (\cosh^2 \sqrt{1/k} - \cosh \sqrt{1/k}y \cosh \sqrt{1/k} \\ + k (\cosh \sqrt{1/k} \cosh \sqrt{1/k}\, y - \cosh^2 \sqrt{1/k}) \\ + (k_0 + k_0^*/k)(y \cosh \sqrt{1/k} \sinh \sqrt{1/k}\, y - \cosh \sqrt{1/k}\, y \sinh \sqrt{1/k}\}\end{array}}{(1/k) \cosh^2 1/k}$$

$$+ \frac{4}{\pi} \sum_{n=0}^{\infty} (-1)^{n+1} e^{s_n t} [A_0((2n+1)^2 \frac{\pi^2}{4} + \frac{1}{k}) + B_0(k_0^* (2n+1)^2 \frac{\pi^2}{4} - 1)]$$

$$\frac{\cos (2n+1) \frac{\pi}{2} y}{(2n+1) \left[(2n+1)^2 \frac{\pi^2}{4} + \frac{1}{k} \right]^2} \quad ...(3.11)$$

Case III Harmonic pressure gradient $-\frac{\partial p}{\partial x} = k_0 \cos wt$.

Equation (3.1) becomes

$$\frac{\partial u}{\partial t} = k_0 \cos wt + \frac{\partial^2 u}{\partial y^2} - k_0^* \frac{\partial^3 u}{\partial t\, \partial y^2} - \frac{u}{k} \quad ...(3.12)$$

Laplace-Transform of this equation is

$$s\bar{u}(y, s) = \frac{k_0 s}{s^2 + w^2} + \frac{d^2 \bar{u}(y, s)}{dy^2} - k_0^* s \frac{d^2 \bar{u}(y, s)}{dy^2} - \frac{\bar{u}(y, s)}{k}$$

$$(1 - k_0^* s) \frac{d^2 \bar{u}(y, s)}{dy^2} - (s + \frac{1}{k}) \bar{u}(y, s) = \frac{k_0^s}{s^2 + w^2}$$

$$\frac{d^2 \bar{u}(y, s)}{dy^2} - \frac{s + 1/k}{1 - K_0^* s} \bar{u}(y, s) = -\frac{k_0 s}{(s^2 + w^2)(1 - k_0^* s)} \quad ...(3.13)$$

Solution of the equation (3.13) under the boundary conditions (3.2) is

$$\bar{u}(y,s) = \frac{k_0 s}{(s^2+w^2)}\left[1 - \frac{\cosh\left(\frac{s+1/k}{1-k_0^* s}\right)^{1/2} y}{\cosh\left(\frac{s+1/k}{1-k_0^* s}\right)^{1/2}}\right]\frac{1}{(s+1/k)}$$

$$= \frac{k_0 s}{(s^2+w^2)(s+1/k)}\left[1 - \frac{\cosh\left(\frac{s+1/k}{1-k_0^* s}\right)^{1/2} y}{\cosh\left(\frac{s+1/k}{1-k_0^* s}\right)^{1/2}}\right] \quad \text{...(3.14)}$$

Using inversion formula, we have

$$u(y,t) = \frac{e^{-iwt} k_0 k\, K_0 k}{2\,(1-iwk)}\left[1 - \frac{\cosh\left(\frac{1/k-iw}{1+k_0^* wi}\right)^{1/2} y}{\cosh\left(\frac{1/k-iw}{1+k_0^* wi}\right)^{1/2}}\right]$$

$$+ \frac{e^{iwt} k_0 k}{2\,(1+iwk)}\left[1 - \frac{\cosh\left(\frac{1/k+iw}{1-k_0^* iw}\right)^{1/2}}{\cosh\left(\frac{1/k+iw}{1-k_0^* iw}\right)^{1/2}}\right]$$

$$+ \frac{4}{\pi}\sum \frac{k_0 s_n e^{s_n t} \cos\,(2n+1)\,\pi/2\,y}{(s_n^2+w^2)\,(k_0^*\,(2n+1)^2\,\pi^2/4-1)\,(2n+1)} \quad \text{...(3.15)}$$

where $$s_n = \frac{(2n+1)^2\,\pi^2/4+1/k}{k_0^*(2n+1)^2\,\pi^2/4-1}$$

and $k_0^*\,(2n+1)^2\,\pi^2/4 < 1,\ s_n t \to 0$ as $t \to \infty$.

4. CONCLUSION AND DISCUSSION

Taking $t = 0$, the obtained velocity prifiles in cases *I, II* and *III* reduce to the velocity profile of steady flow of Walters liquid *B* through porous stratum with constant pressure gradient.

Assuming $t=0$ and $k_0^*=0$, the obtained velocity profiles reduce to the velocity profile obtained by Manvi and Ahmadi [1] for viscous fluid in steady motion.

For an infinite lapse of time, it is observed that k_0^* has no effect on flow velocity though the effect of permeability is still present. From the relation (3.7), it is quite clear that the effect of viscoelasticity is very small and is not significant physically.

The graph is to show the variation of u given by the equation (3.7) with porousity k of media.

As k increases, the velocity u also increases.

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