Jñānābha, Vol.26,1996

# SOME ABELIAN THEOREMS FOR DISTRIBUTIONAL HANKEL-CLIFFORD TRANSFORMATION

*By*

**Jay Ram Mahto**
*Department of Mathematics,*
*P.P.K. College, Bundu (Ranchi), Bihar*

and

**Anil Kumar Mahato**
*Department of Mathematics,*
*Marwari College, Ranchi-834001, India*

*(Received : April 26, 1993)*

## ABSTRACT

In this paper an extension of the Hankel-Clifford transformation

$$F(s) = s^{\mu} \int_0^{\infty} (st)^{-\mu/2} J_{\mu} (2\sqrt{st})\, f(t)\, dt$$

to generalized functions is provided. Some Abelian theorem for the distributional Hankel-Clifford transform are proved.

**1. INTRODUCTION :** The integral transform

$$F(s) = s^{\mu} \int_0^{\infty} (st)^{-\mu/2} J_{\mu} (2\sqrt{st})\, f(t)\, dt \qquad ...(1.1)$$

where $J_{\mu}$ is the bessel function of the first kind of order $\mu$, was studied by Betancor [1]. (1.1) is called as Hankel-Clifford transform.

In this paper, we shall relate the behaviour of generating function $F(s)$ as $s$ approaches zero or infinity to the behaviour of determining function $f(t)$ as $t$ approaches zero or infinity, respectively. Theorems of this nature are called Abelian theorems. In section 2, we construct a testing function space and its dual for Hankel-Clifford transform. We prove initial and final value theorems in section 3 and 4 respectively.

**NOTATIONS AND TERMINOLOGY** The notations and terminology of this work will follow that of [4] and [10]. Let $I$ denotes the interval $(0, \infty)$. $D(I)$ is the space of infinitely smooth functions defined over $I$ whose supports are compact subset of $I$. $D'(I)$ is Schwartz's dual space of distributions on $I$. $E(I)$ is the space of smooth functions on $I$, and $E'(I)$, the dual space of distributions having compact supports with respect to $I$. We assign to all these spaces their customary topologies [6, pp. 88-90].

## 2. THE TESTING FUNCTIONS SPACE $H_{c,d}$ AND ITS DUAL $H'_{c,d}$

We define the differential operator

$$S_{\mu,t}\,\phi(t) = Dt^{-\mu+1}\,Dt^{-\mu}\,\phi(t) : D = \frac{\partial}{\partial t} \qquad ...(2.1)$$

Let us define the seminorms $\delta_{c,d,k}$; $k = 0, 1, 2, ...$, on a smooth function $\phi(t)$ defined on $0 < t < \infty$, by

$$\delta_{c,d,k}(\phi) = \sup_{0<t<\infty} |\lambda_{c,d}(t)\, t^{\mu/2}\, s^{k}_{\mu,t}\,\phi(t)|,$$

where $\mu$ is a complex number with Re $\mu > 0$ and

$$\lambda_{c,d}(t) = t^{c}, \quad 1 \le t < \infty$$
$$= t^{d}, \quad 0 < t < 1.$$

$H_{c,d}$ is the space of smooth functions $\phi(t)$ on $0 < t < \infty$ for which $\delta_{c,d,k}(\phi)$ is finite for all $k = 0, 1, 2, ....$

$H_{c,d}$ is a complete countably multinormed space $H'_{c,d}$ denotes the dual of $H_{c,d}$. If $f \in H'_{c,d}$ the Hankel-Clifford transform $F$ of $f$ is defined by

$$F(s) = \langle f(t), S^{\mu}(st)^{-\mu/2} J_{\mu}(2\sqrt{st})\rangle \qquad ...(2.3)$$

for any complex $s$ not lying on the negative real axis and Re $\mu > 0$.

**Theorem 2.1** *The kernel* $k(st) = s^{\mu}(st)^{-\mu/2} J_{\mu}(2\sqrt{st})$ *is a member of* $H_{c,d}$ *if* $c < \frac{1}{4}$ *and* $d > 0$.

**Proof :** $k(st)$ is a smooth function and is a member of $H_{c,d}$ if and only if

$$\sup_{0<t<\infty} |\lambda_{c,d}(t)\, t^{\mu/2}\, S^{k}_{\mu,t}\{K(st)\}| < \infty.$$

From [10, p.154], we have

$$D_x\, x^{\mu} J_{\mu}(xy) = yx^{\mu} J_{\mu-1}(xy)$$

$$D_x\, x^{-\mu} J_{\mu}(xy) = -yx^{-\mu} J_{\mu+1}(xy)$$

Hence

$$\sup_{0<t<\infty} |\lambda_{c,d}(t)\, t^{\mu/2} S^k_{\mu,t}\{K(st)\}|$$

$$= \sup_{0<t<\infty} |\lambda_{c,d}(t)\, s^{\mu/2+2k} J_\mu(2\sqrt{st})|$$

We have $J_\mu(x) = O(x^\mu)$ as $x \to 0$ and $\sqrt{x}\, J_\mu(x) = O(1)$ as $x \to \infty$.

Thus for $0 < t \le 1$,

$$\sup_{0<t\le 1} |\lambda_{c,d}(t)\, t^{\mu/2} S^k_{\mu,t}\{k(st)\}|$$

$$= \sup_{0<t\le 1} |t^d s^{\mu/2+2k} J_\mu(2\sqrt{st})|$$

which tends to zero as $t \to 0$, $s$ fixed and $d > 0$.

For $|t| \to \infty$,

$$\sup_{1<t<\infty} |\lambda_{c,d}(t)\, t^{\mu/2} S^k_{\mu,t}\{K(st)\}|$$

$$= \sup_{1<t<\infty} |t^{c-1/4}\, 2^{-1/2}\, s^{\mu/2+2k-1/4}\, (2\sqrt{st})^{1/2} J_\mu(2\sqrt{st})|$$

which is bounded for $|t| \to \infty$ if $c < 1/4$.

Thus $K(st) \in H_{c,d}$ for $c < 1/4$ and $d > 0$.

**Theorem 2.2** *If $f \varepsilon H'_{c,d}$ and $F(s)$ is defined by (2.3), then*

$$F^k(s) = \left\langle f(t), \frac{\partial}{\partial s^k}[s^\mu (st)^{-\mu/2} J_\mu(2\sqrt{st})]\right\rangle \qquad \ldots(2.5)$$

**Proof :** The right hand side of (2.3) has a sense as the application of $f(t)\, \varepsilon\, H_{c,d}$ since $K(st)\, \varepsilon\, H_{c,d}$. For a similar reason, (2.5) also has meaning. (2.5) can be proved by using Cauchy's integral formula. The proof is very similar to that followed in [10, Theorem 8.3.1] and is therefore omitted.

**Theorem 2.3** If $F(s)$ is defined by (2.3)

$$F(s) = O(1),\ s \to \infty$$

$$F(s) = O(s^{-1}),\ s \to 0$$

**Proof :** Using boundedness property of a generalized function, we have

$$|F(s)| = |\langle f, \phi\rangle|$$

$$\le C \max_{0\le k\le r} \sup_{0<t<\infty} |\lambda_{c,d}(t)\, t^{\mu/2} S^k_{\mu,t}\, \phi(t)|$$

for appropriate constants $C$ and $r$.

Therefore

$$|F(s)| \leq C \max_{0 \leq k \leq r} \sup_{0<t<\infty} |\lambda_{c,d}(t)\, t^{\mu/2} S^{k}_{\mu,t} [s^{\mu}(st)^{-\mu/2} J_{\mu}(2\sqrt{st})]|$$

$$\leq C \max_{0 \leq k \leq r} \sup_{0<t<\infty} |\lambda_{c,d}(t)\, s^{\mu/2+2k} J_{\mu}(2\sqrt{st})|$$

Using the result (2.4), we have

$$|s\,F(s)| \to 0 \text{ as } s \to 0+$$

$$|F(s)| \to 0 \text{ as } s \to \infty.$$

From this our theorem is proved. Theorem 2.3 can be easily generalized to the following :

**Theorem 2.4** If $F(s)$ is defined by (2.3) then

$$|F^{k}(s)| = O(s^{-k-1});\ s \to 0+$$

$$|F^{k}(s)| = O(s^{-k});\ s \to \infty.$$

## 3. AN INITIAL-VALUE THEOREM FOR HANKEL-CLIFFORD TRANSFORMATION

**Theorem 3.1** *Let*

(*i*) $f(t) \to 0$ *as* $t \to \infty$

(*ii*) $f(t)/t^{n}$ *is absolutely continuous on* $0 \leq t < \infty$ *where* $\eta$ *is a real number and if there exists a complex number* $\alpha$ *such that*

$$\lim_{t \to 0+} \frac{f(t)}{t^{\eta}} = \alpha \qquad \ldots(3.1)$$

*then*

$$\lim_{s \to 0} C s^{\eta+1-\mu} F(s) = \alpha$$

*where* $F(s)$ *is given by (2.3) and* $C = \dfrac{\Gamma(\mu-\eta+1)}{\Gamma(\eta+1)}$ *provided* $\eta + 1 > 0$, $\mu > \eta - 1$ *and* $\eta > \frac{3}{4} - \frac{\mu}{2}$.

**Proof :** By our hypothesis, $f(t) = O(t^{\eta})$ as $t \to 0+$. Moreover, the transform (2.3) exists for every positive value of $\mu$ and $\eta$.

From [2, p.326], we have

$$\int_0^{\infty} x^{s-1} \sqrt{x}\, J_{\upsilon}(x)\, dx = \frac{2^{2-\frac{1}{2}}\, \Gamma(s/2 + \upsilon/2 + 1/4)}{\Gamma(\upsilon/2 - s/2 + 3/4)}$$

By using this, we get

$$\int_0^\infty t^\eta s^\mu (st)^{-\mu/2} J_\mu (2\sqrt{st})\, dt = s^{\mu-\eta-1} \frac{\Gamma(\eta+1)}{\Gamma(\mu-n+1)} \qquad ...(3.2)$$

**Assuming that $y > 0$, we may write**

$$\left| s^{\eta+1-\mu} F(s) - \frac{\alpha}{C} \right|$$

$$= \left| s^{\eta+1-\mu} \int_0^\infty s^\mu (st)^{-\mu/2} J_\mu (2\sqrt{st})\, f(t)\, dt - \frac{\alpha}{C} \right|$$

$$= \left| s^{\eta+1-\mu} \left\{ \int_0^\infty s^\mu (st)^{-\mu/2} J_\mu (2\sqrt{st})\, f(t)\, dt - \alpha \int_0^\infty t^n s^{\mu/2} (st)^{-\mu/2} J_\mu (2\sqrt{st})\, dt \right\} \right|$$

$$= \left| s^{\eta+1-\mu} \int_0^\infty [f(t) - \alpha t^\eta]\, s^\mu (st) (st)^{-\mu/2} J_\mu (2\sqrt{st})\, dt \right|$$

$$\le \left| s^{\eta+1-\mu} \int_0^y [f(t) - \alpha t^\eta]\, s^\mu (st)^{-\mu/2} J_\mu (2\sqrt{st})\, dt \right|$$

$$+ \left| s^{\eta+1-\mu} \int_y^\infty [f(t) - \alpha t^\eta]\, s^\mu (st)^{-\mu/2} J_\mu (2\sqrt{st})\, dt \right|$$

$$= I_1 + I_2.$$

**Now**

$$I_1 \le s^{\eta+1-\mu} \sup_{0 < t \le y} \left| \frac{f(t)}{t^\eta} - \alpha \right| \left| \int_0^\infty t^\eta s^\mu (st)^{-\mu/2} J_\mu (2\sqrt{st})\, dt \right|$$

$$= s^{\eta+1-\mu} \sup_{0 \le t \le y} \left| \frac{f(t)}{t^\eta} - \alpha \right| \left| \frac{s^{\mu-\eta-1}}{C} \right| \qquad \text{by (3.2).}$$

**We choose $y$ so small that** $\left| \frac{f(t)}{t^\eta} < \alpha \right| < \frac{\varepsilon}{C}$ in $0 \le t \le y$ for $\varepsilon > 0$.

**Hence $I_1 \to 0$.** ...(3.3)

**Having fixed $y$ in this way and using (2.4).**

$$I_2 \le M s^{\eta+1-\mu/2-1/4} \left| \int_y^\infty t^{\eta-\mu/2-1/4} \left[ \frac{f(t)}{t^\eta} - \alpha \right] dt \right|$$

**for some constant $M$. Since $f(t)/t^\eta$ is a bounded quantity in $y \le t < \infty$ for some constant $M_1$.**

$$I_2 \le M_1 s^{\eta+3/4-\mu/2} \frac{y^{\eta-\mu/2+3/4}}{\eta-\mu/2+3/4} \to 0 \text{ as } s \to 0. \qquad ...(3.4)$$

From (3.2) and (3.3) the result follows.

To extend the preceding results to the space $H_{c,d}$ we require the notion of the value of the distribution at a point. This concept is introduced by Lojasiewicz [3].

**Defintion 3.1. Lojasiewicz [3].** *Let T be a distribution defined in a neighbourhood of a point $x_0$. We say that T has a value C at $x_0$ i.e. $T(x_0) = C$ if the distributional limit $\lim_{\lambda \to 0+} T(x_0 + \lambda x)$ exists in a neighbourhood of $x_0$ and if it is a constant function C.*

**Theorem 3.2** *(distributional initial value Abelian theorem), if*

*(i)* $f(t) \varepsilon H'_{c,d}$

*(ii)* $f(t)/t^{\eta} \to \alpha$ *as* $t \to 0$ *in the sence of Logasiewicz, then*

$$\lim_{s \to \infty} C\, s^{\eta + 1 - \mu} F(s) = \alpha,$$

*where F(s) and C are defined in Theorem 3.1, provided* $\eta + 1 > 0$ *and* $\mu > r$

**Proof :** Let us consider

$$s^{\eta + 1 - \mu} F(s) - \frac{\alpha}{C} = s^{\eta + 1 - \mu} < f(t) - \alpha t^{\eta}, s^{\mu} (st)^{-\mu/2} J_{\mu} (2\sqrt{st})>$$

$$= s^{\eta + 1 - \mu} G(s) \qquad \text{using (3.2)}$$

where $G(s) = <f(t) - \alpha t^{\eta}, s^{\mu} (st)^{-\mu/2} J_{\mu} (2\sqrt{st})>$.

Now from the boundedness property of generalized functions, there exists a positive constant $M$ and non-negative integer $r$ such that

$$\left| G(s) \right| \leq M \max_{0 \leq k \leq r} \sup_{0 < t < \infty} \left| \delta_{c,d,k} [K(st)] \right|$$

for a suitably chosen constant $M_1$. Now from (2.2) and (2.4), we have

$$|s^{\eta + 1 - \mu} G(s)| \leq \sup_{0 < t < \infty} M_2 t^{d + \mu/2} s^{\eta + 1 + 2k}$$

$$+ \sup_{1 \leq t < \infty} M_3 t^{C - 1/4} s^{\eta - \mu/2 + 2k + 3/4}$$

$$= M_2 + M_3 s^{\eta - \mu/2 + 2k + 3/4}$$

$$= M_4 s^{\eta - \mu/2 + 2k + 3/4}$$

for some suitably chosen constants $M_2$, $M_3$ and $M_4$. Since

$$\eta - \frac{\mu}{2} + \frac{3}{4} > 0, s^{\eta + 1 - \mu} G(s) \to 0 \text{ as } s \to \infty.$$

This establishes the theorem.

## 4. FINAL VALUE THEOREM FOR HANKEL-CLIFFORD TRANSFORMATION

**Theorem 4.1** *Let $f(t)$ be a measurable function on $0 < t < \infty$ and if there exist a real $\eta$ and a complex $\alpha$ such that*

$$(i) \lim_{t \to \infty} f(t)/t^{\eta} = \alpha \qquad \ldots(4.1)$$

*and* $(ii)$ *$f(t)/t^{\eta}$ is bounded on $0 \le t \le y$ for all $y > 0$.*

*Then*
$$\lim_{s \to \infty} s^{\eta + 1 - \mu}\, CF(s) = \alpha,$$

*where $F(s)$ and $C$ are given as in Theorem 3.1; provided $\eta + 1 > 0$ and $\mu > \eta - 1$.*

**Proof :** (4.1) indicates that $f(t)$ is a function of slow growth and its Hankel-Clifford transform converges for $s > 0$. Proceeding as in the proof of Theorem 3.1, we have

$$| s^{\eta + 1 - \mu} F(s) - \frac{\alpha}{C} |$$

$$= | s^{\eta + 1 - \mu} \int_0^{\infty} s^{\mu} (st)^{-\mu/2} J_{\mu} (2\sqrt{st})\, f(t)\, dt -$$
$$\alpha\, s^{\eta + 1 - \mu} \int_0^{\infty} s^{\mu} (st)^{-\mu/2} J_{\mu} (2\sqrt{st})\, dt |$$

$$= | s^{\eta + 1 - \mu} \int_0^{\infty} s^{\mu} (st)^{-\mu/2} J_{\mu} (2\sqrt{st})\, [f(t) - \alpha t^{\eta}]\, dt |$$

$$\le \int_0^{y} | s^{\eta + 1 - \mu} s^{\mu} (st)^{-\mu/2} J_{\mu} (2st)\, [f(t) - \alpha t^{\eta}] |\, dt$$
$$+ \int_y^{\infty} | s^{\eta + 1 - \mu} s^{\mu} (st)^{-\mu/2} J_{\mu} (2st)\, [f(t) - \alpha t^{\eta}] |\, dt$$

$$= J_1 + J_2.$$

From condition $(i)$ of the theorem, we can find a small positive number $\in$ such that for $y$ large enough

$$\sup_{y \le t < \infty} | f(t)/t^{\eta} - \alpha | \le\, \in C.$$

Hence we have

$$|J_2| \le\, \in C \int_0^{\infty} s^{\eta + 1 - \mu} s^{\mu} (st)^{-\mu/2} J_{\mu} (2\sqrt{st})\, t^{\eta}\, dt$$
$$= \in \qquad \text{by (3.2)}$$

hence $\quad |J_2| \le\, \in$

Therefore $|J_2| \to 0$ as $\epsilon \to 0$.

Now having fixed $y$ as above, we have

$$|J_1| \leq \sup_{0 \leq t \leq y} |f(t)/t^{\eta - \alpha}| \int_0^y |s^{\eta + 1 - \mu} s^{\mu} (st)^{-\mu/2} J_{\mu} (2\sqrt{st}) \;^{n}| \, dt.$$

Due to conditions (ii) and (2.4), we have

$|J_1| \to 0$ as $s -> \infty$ since $\eta + 1 > 0$ and $\mu - \eta + 1 > 0$.

This completes the proof.

In extending this theorem to distributions we shall need the following result (Zemanian [Section 3.3]).

If $f \varepsilon E'(I)$ there exists a constant $C$ and a non-negative integer $r$ such that for every $\phi \varepsilon D(I)$.

$$|\langle f, \phi \rangle \leq C \sup_{0 < t < \infty} |D_t^r \phi(t)| \qquad ...(4.2)$$

where $D_t \equiv \frac{d}{dt}$.

**Theorem 4.2** (*distributional final value/Abelian theorem*). *If*

*(i)* $f \varepsilon H'_{c,d}$ *and* $f$ *can be decomposed into* $f = f_1 + f_2$ *where* $f_1$ *is an ordinary function and* $f_2 \varepsilon E'(I)$, $f_1$ *satisfies the hypothesis of Theorem 4.1.*

*(ii)* $f(t)/t^{n} \to \alpha$ *as* $t \to \infty$ *in the sense of Lojasiewiez.*

*(iii)* $0 < \eta + 1 < \mu$;

*(iv)* $F(s)$ *is the distributional Hankel-Clifford transform of f, then*

$$\lim_{s \to \infty} s^{\eta + 1 - \mu} F(s) = \lim_{t \to \infty} \frac{f(t)}{t^{\eta} C}$$

*where C is defined in Theorem 3.1.*

**Proof :** $F(s) = F_1(s) + F_2(s)$ and

$$F_2(s) = \langle f_2(t), s^{\mu} (st)^{-\mu/2} J_{\mu}(2\sqrt{st}) \rangle$$

By theorem 2.2, is smooth function and is of slow growth as $s \to \infty$. Let $g(t) \varepsilon D(I)$ be ideticaIly equal to one on a neighbourhood of support of $f_2$. From (4.2)

$$|F_2(s)| = |\langle f_2(t), g(t) s^{\mu} (st)^{-\mu/2} J_{\mu} (2\sqrt{st}) \rangle|$$

$$\leq C_1 \sup_{0 < t < \infty} |D_t^r [g(t) s^{\mu} (st)^{-\mu/2} J_{\mu} (2\sqrt{st})]|$$

$$= C_1 \sup_{0<t<\infty} \sum_{\nu=0}^{r} \binom{r}{\nu} |D^{r-\nu} g(t)| \, |D_t^{\nu} s^{\mu} (st)^{-\mu/2} J_\mu (2\sqrt{st})|$$

$$= C_1 \sup_{0<t<\infty} \sum_{\upsilon=0}^{r} \binom{r}{\nu} |D^{r-\upsilon} g(t)| \, |(-1)^{\upsilon} s^{\mu+\upsilon}$$

$$(st)^{-(\mu+\upsilon)/2} J_{\mu+\upsilon} (2\sqrt{st})|$$

From (2.4) it is easy to see that

$$|(-1)^{\nu} s^{\mu+\nu} (st)^{-(\mu+\nu)/2} J_\mu (2\sqrt{st})|$$

is bounded on $0 < t < \infty$.

Hence for some constant $C_2$,

$$|F_2(s)| \le C_2$$

or

$$|s^{\eta+1-\mu} F_2(s)| \le C_2 \, s^{\eta+1-\mu}$$

Under condition (*iii*),

$$|s^{\eta+1-\mu} F_2(s)| \to 0 \text{ as } s \to \infty.$$

Then

$$\lim_{s\to\infty} s^{\eta+1-\mu} F(s)$$

$$= \lim_{s\to\infty} s^{\eta+1-\mu} F_1(s) + \lim_{s\to\infty} s^{\eta+1-\mu} F_2(s)$$

$$= \lim_{s\to\infty} s^{\eta+1-\mu} F_1(s).$$

Since $f_1$ is an ordinary function which satisfies the hypothesis of Theorem 4.1 the required result follows.

## REFERENCES


[1] J.J. Betancor, The Hankel-Clifford Transformation on certain spaces of ultradistribution, *Indian J. Pure Appl. Math.* **20** (6) (1989), 583-603.

[2] A. Erdely, *Tables of Integral Transforms Vol. I,* McGraw - Hill Book Company, Inc. N.Y., 1954.

[3] S. Lojasiewiez, Sur la valcur et la limite d' une distribution en un point, *Studia Math.* **16** (1957), 1-36.

[4] J.R. Mahto and A.K. Mahato, On the Hankel-Clifford Transformation (Communicated for publication in *Bulletin of Institute of Mathematics, Academic Sinica, China*).

[5] O.P. Misra, Some Abelian theorems for distributional Stieltjes transformation, *J. Math. Anal. Appl.* **39**, No. 3 (1972), 590-599.

[6] L. Schwartz, *Theorie des Distributions,* Herman, Paris, 1966.

[7] A.K. Tiwari and P.S. Koranne, Abelian theorems for a distribution generalized Stieltjes transform, *Indian J. Pure Appl. Math.* **16**(4) (19[illegible]), 383-394.

[8] A.H. Zemanian, Some Abelian theorems for the distributional Hankel and *K*-Transformations, *SIAM J. Appl. Math.* **14,** No. 6 (1966), 1255-[illegible].

[9] A.H. Zemanian, *Distribution Theory and Transform Analysis,* McGraw Hill, New York, 1965.

[10] A.H. Zemanian, *Generalized Integral Transformations Interscience,* New York, 1968.