SOME ABELIAN THEOREMS FOR DISTRIBUTIONAL HANKEL-CLIFFORD TRANSFORMATION

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ABSTRACT

In this paper an extension of the Hankel-Clifford transformation

\[ F(s) = s^\mu \int_0^\infty (st)^{-\mu/2} J_\mu(2\sqrt{st}) f(t) \, dt \]

... (1.1)

to generalized functions is provided. Some Abelian theorem for the distributional Hankel-Clifford transform are proved.

1. INTRODUCTION: The integral transform

\[ F(s) = s^\mu \int_0^\infty (st)^{-\mu/2} J_\mu(2\sqrt{st}) f(t) \, dt \] ... (1.1)

where \( J_\mu \) is the bessel function of the first kind of order \( \mu \), was studied by Betancor [1]. (1.1) is called as Hankel-Clifford transform.

In this paper, we shall relate the behaviour of generating function \( F(s) \) as \( s \) approaches zero or infinity to the behaviour of determining function \( f(t) \) as \( t \) approaches zero or infinity, respectively. Theorems of this nature are called Abelian theorems. In section 2, we construct a testing function space and its dual for Hankel-Clifford transform. We prove initial and final value theorems in section 3 and 4 respectively.

NOTATIONS AND TERMINOLOGY The notations and terminology of this work will follow that of [4] and [10]. Let \( I \) denotes the interval \((0, \infty)\). \( D(I) \) is the space of infinitely smooth functions defined
over $I$ whose supports are compact subset of $I$. $D'(I)$ is Schwartz's dual space of distributions on $I$. $E(I)$ is the space of smooth functions on $I$, and $E'(I)$, the dual space of distributions having compact supports with respect to $I$. We assign to all these spaces their customary topologies [6, pp. 88-90].

2. THE TESTING FUNCTIONS SPACE $H_{c,d}$ AND ITS DUAL $H_{c,d} '$

We define the differential operator

$$S_{\mu, \phi(t)} = D t^{-\mu + 1} D t^{-\mu} \phi(t) : D = \frac{d}{dt} \quad \ldots (2.1)$$

Let us define the seminorms $\delta_{c,d,k}; k = 0, 1, 2, \ldots$, on a smooth function $\phi(t)$ defined on $0 < t < \infty$, by

$$\delta_{c,d,k} (\phi) = \sup_{0 < t < \infty} |\lambda_{c,d} (t) t^{\mu/2} \delta_{\mu, t} \phi(t)|,$$

where $\mu$ is a complex number with $\Re \mu > 0$ and

$$\lambda_{c,d} (t) = \begin{cases} t^c, & 1 \leq t < \infty \\ t^d, & 0 < t < 1. \end{cases}$$

$H_{c,d}$ is the space of smooth functions $\phi(t)$ on $0 < t < \infty$ for which $\delta_{c,d,k} (\phi)$ is finite for all $k = 0, 1, 2, \ldots$.

$H_{c,d}$ is a complete countably multinormed space $H_{c,d} '$ denotes the dual of $H_{c,d}$. If $f \in H_{c,d}$ the Hankel-Clifford transform $F$ of $f$ is defined by

$$F(s) = \langle f(t), s^\mu (st)^{-\mu/2} J_\mu (2\sqrt{st}) > \quad \ldots (2.3)$$

for any complex $s$ not lying on the negative real axis and $\Re \mu > 0$.

Theorem 2.1 The kernel $k(st) = s^\mu (st)^{-\mu/2} J_\mu (2\sqrt{st})$ is a member of $H_{c,d}$ if $\epsilon < \frac{1}{2}$ and $d > 0$.

Proof: $k(st)$ is a smooth function and is a member of $H_{c,d}$ if and only if

$$\sup_{0 < t < \infty} |\lambda_{c,d} (t) t^{\mu/2} \delta_{\mu, t} [K(st)]| < \infty.$$

From [10, p.154], we have

$$D_x x^\mu J_\mu (xy) = yx^\mu J_{\mu - 1} (xy)$$

$$D_x x^{-\mu} J_\mu (xy) = -yx^{-\mu} J_{\mu + 1} (xy)$$
Hence

$$\sup_{0 < t < \infty} |\lambda_{c,d} (t) t^{\mu/2} S_{\mu}^k [K(st)]|$$

$$= \sup_{0 < t < \infty} |\lambda_{c,d} (t) s^{\mu/2} + 2k J_{\mu} (2\sqrt{s t})|$$

We have $J_{\mu}(x) = O(x^\mu)$ as $x \to 0$ and $\sqrt{x} J_{\mu}(x) = O(1)$ as $x \to \infty$.

Thus for $0 < t \leq 1$,

$$\sup_{0 < t \leq 1} |\lambda_{c,d} (t) t^{\mu/2} S_{\mu}^k [k(st)]|$$

$$= \sup_{0 < t \leq 1} |t^d s^{\mu/2} + 2k J_{\mu} (2\sqrt{s t})|$$

which tends to zero as $t \to 0$, $s$ fixed and $d > 0$.

For $|t| \to \infty$,

$$\sup_{1 < t < \infty} |\lambda_{c,d} (t) t^{\mu/2} S_{\mu}^k [K(st)]|$$

$$= \sup_{1 < t < \infty} |t^{c - 1/4} 2^{1/2} s^{\mu/2} + 2k - 1/4 (2\sqrt{s t})^{1/2} J_{\mu} (2\sqrt{s t})|$$

which is bounded for $|t| \to \infty$ if $c < 1/4$.

Thus $K(st) \in H_{c,d}$ for $c < 1/4$ and $d > 0$.

**Theorem 2.2** If $f \in H_{c,d}$ and $F(s)$ is defined by (2.3), then

$$F^k(s) = \langle f(t), \frac{\partial}{\partial s^k} [s^\mu (st)^{-\mu/2} J_{\mu} (2\sqrt{s t})]\rangle \quad \ldots (2.5)$$

**Proof**: The right hand side of (2.3) has a sense as the application of $f(t) \in H_{c,d}$ since $K(st) \in H_{c,d}$. For a similar reason, (2.5) also has meaning. (2.5) can be proved by using Cauchy's integral formula. The proof is very similar to that followed in [10, Theorem 8.3.1] and is therefore omitted.

**Theorem 2.3** If $F(s)$ is defined by (2.3)

$$F(s) = O(1), s \to \infty$$

$$F(s) = O(s^{-1}), s \to 0$$

**Proof**: Using boundedness property of a generalized function, we have

$$|F(s)| = |\langle f, \phi \rangle|$$

$$\leq C \max_{0 \leq k \leq r} \sup_{0 < t < \infty} |\lambda_{c,d} (t) t^{\mu/2} S_{\mu}^k \phi(t)|$$
for appropriate constants $C$ and $r$.

Therefore

$$|F(s)| \leq C \max_{0 \leq k \leq r} \sup_{0 < t < \infty} |\lambda_{c,d}(t) t^{\mu/2} S_{\mu,t}^k [s^{\mu/2} J_{\mu}(2\sqrt{s}t)]|$$

$$\leq C \max_{0 \leq k \leq r} \sup_{0 < t < \infty} |\lambda_{c,d}(t) s^{\mu/2 + 2k} J_{\mu}(2\sqrt{s}t)|$$

Using the result (2.4), we have

$$|s F(s)| \rightarrow 0 \text{ as } s \rightarrow 0^+$$

$$|F(s)| \rightarrow 0 \text{ as } s \rightarrow \infty.$$  

From this our theorem is proved. Theorem 2.3 can be easily generalized to the following:

**Theorem 2.4** If $F(s)$ is defined by (2.3) then

$$|F^k(s)| = O(s^{-k-1}); \quad s \rightarrow 0^+$$

$$|F^k(s)| = O(s^{-k}); \quad s \rightarrow \infty.$$  

3. AN INITIAL-VALUE THEOREM FOR HANKEL-CLIFFORD TRANSFORMATION

**Theorem 3.1** Let

(i) $f(t) \rightarrow 0$ as $t \rightarrow \infty$

(ii) $f(t)/t^n$ is absolutely continuous on $0 \leq t < \infty$ where $\eta$ is a real number and if there exists a complex number $\alpha$ such that

$$\lim_{t \rightarrow 0^+} \frac{f(t)}{t^n} = \alpha \quad \ldots (3.1)$$

then

$$\lim_{s \rightarrow 0} C s^{\eta + 1 - \mu} F(s) = \alpha$$

where $F(s)$ is given by (2.3) and $C = \frac{\Gamma(\mu - \eta + 1)}{\Gamma(\eta + 1)}$ provided $\eta + 1 > 0,$ $\mu > \eta - 1$ and $\eta > \frac{3}{4} - \frac{\mu}{2}$

**Proof:** By our hypothesis, $f(t) = O(t^n)$ as $t \rightarrow 0^+.$ Moreover, the transform (2.3) exists for every positive value of $\mu$ and $\eta.$

From [2, p.326], we have

$$\int_0^\infty x^{s-1} \sqrt{x} J_v(x) \, dx = \frac{2^{2-v} \Gamma(\nu/2 + 1/4)}{\Gamma(\nu/2 - s/2 + 3/4)}$$

By using this, we get
\[
\int_0^t t^n s^\mu (st)^{-\mu/2} J_\mu (2\sqrt{st}) \, dt = s^\mu - \eta - 1 \frac{\Gamma(\eta + 1)}{\Gamma(\mu - \eta + 1)} \quad \text{...}(3.2)
\]

Assuming that \( y > 0 \), we may write

\[
\begin{align*}
&\left| s^{\eta+1-\mu} F(s) - \frac{\alpha}{C} \right| \\
&= \left| s^{\eta+1-\mu} \int_0^\infty s^\mu (st)^{-\mu/2} J_\mu (2\sqrt{st}) f(t) \, dt - \frac{\alpha}{C} \right| \\
&= \left| s^{\eta+1-\mu} \left\{ \int_0^\infty s^\mu (st)^{-\mu/2} J_\mu (2\sqrt{st}) f(t) \, dt \\
&\quad - \alpha \int_0^\infty t^n s^\mu (st)^{-\mu/2} J_\mu (2\sqrt{st}) \, dt \right\} \right| \\
&= \left| s^{\eta+1-\mu} \int_0^\infty [f(t) - \alpha t^n] s^\mu (st)^{-\mu/2} J_\mu (2\sqrt{st}) \, dt \right| \\
&\leq \left| s^{\eta+1-\mu} \int_0^y [f(t) - \alpha t^n] s^\mu (st)^{-\mu/2} J_\mu (2\sqrt{st}) \, dt \right| \\
&\quad + \left| s^{\eta+1-\mu} \int_y^\infty [f(t) - \alpha t^n] s^\mu (st)^{-\mu/2} J_\mu (2\sqrt{st}) \, dt \right| \\
&= I_1 + I_2.
\end{align*}
\]

Now

\[
I_1 \leq s^{\eta+1-\mu} \sup_{0 < t \leq y} \left| \frac{f(t)}{t^n} - \alpha \right| \left| \int_0^\infty t^n s^\mu (st)^{-\mu/2} J_\mu (2\sqrt{st}) \, dt \right|
\]

\[
= s^{\eta+1-\mu} \sup_{0 < t \leq y} \left| \frac{f(t)}{t^n} - \alpha \right| \left| \frac{s^\mu - \eta - 1}{C} \right| \quad \text{by (3.2)}.
\]

We choose \( y \) so small that \( \left| \frac{f(t)}{t^n} - \alpha \right| < \frac{\varepsilon}{C} \) in \( 0 \leq t \leq y \) for \( \varepsilon > 0 \).

Hence \( I_1 \to 0 \). \quad \text{...}(3.3)

Having fixed \( y \) in this way and using (2.4),

\[
I_2 \leq M s^{\eta+1-\mu/2-1/4} \int_y^\infty t^{\eta-\mu/2-1/4} \left| \frac{f(t)}{t^n} - \alpha \right| \, dt
\]

for some constant \( M \). Since \( f(t)/t^n \) is a bounded quantity in \( y \leq t < \infty \) for some constant \( M_1 \).

\[
I_2 \leq M_1 s^{\eta+3/4-\mu/2} \frac{\gamma^{\eta-\mu/2+3/4}}{\eta-\mu/2+3/4} \to 0 \quad \text{as} \ s \to 0. \quad \text{...}(3.4)
\]
From (3.2) and (3.3) the result follows.

To extend the preceding results to the space $H_{c,d}$ we require the notion of the value of the distribution at a point. This concept is introduced by Lojasiewicz [3].

**Definition 3.1. Lojasiewicz [3].** Let $T$ be a distribution defined in a neighbourhood of a point $x_0$. We say that $T$ has a value $C$ at $x_0$ i.e. $T(x_0) = C$ if the distributional limit $\lim_{\lambda \to 0^+} T(x_0 + \lambda x)$ exists in a neighbourhood of $x_0$ and if it is a constant function $C$.

**Theorem 3.2** (distributional initial value Abelian theorem), if

(i) $f(t) \in H'_{c,d}$

(ii) $f(t)/t^n \to \alpha$ as $t \to 0$ in the sense of Logasiewicz, then

$$\lim_{s \to \infty} C s^{\eta + 1 - \mu} F(s) = \alpha,$$

where $F(s)$ and $C$ are defined in Theorem 3.1, provided $\eta + 1 > 0$ and $\mu > r$

**Proof:** Let us consider

$$s^{\eta + 1 - \mu} F(s) - \frac{\alpha}{C} = s^{\eta + 1 - \mu} <f(t) - \alpha t^n, s^{\mu} (st)^{-\mu/2} J_{\mu} (2\sqrt{st})>, \quad s^{\eta + 1 - \mu} G(s)$$

where $G(s) = <f(t) - \alpha t^n, s^{\mu} (st)^{-\mu/2} J_{\mu} (2\sqrt{st})>$.

Now from the boundedness property of generalized functions, there exists a positive constant $M$ and non-negative integer $r$ such that

$$\left| \frac{G(s)}{s^{\eta + 1 - \mu}} \right| \leq M \max_{0 \leq k \leq r} \sup_{0 < t < \infty} \left| \delta_{c,d,k} \left[ K(st) \right] \right|$$

for a suitably chosen constant $M_1$. Now from (2.2) and (2.4), we have

$$\left| s^{\eta + 1 - \mu} G(s) \right| \leq \sup_{0 < t < \infty} \left( M_2 t^{\mu/2} s^{\eta + 1 + 2k} + \sup_{1 \leq t < \infty} M_3 t^{1/4} s^{-\mu/2 + 2k + 3/4} \right)$$

$$= M_2 + M_3 s^{-\mu/2 + 2k + 3/4}$$

for some suitably chosen constants $M_2$, $M_3$ and $M_4$. Since

$$\eta - \mu > 0, s^{\eta + 1 - \mu} G(s) \to 0 \text{ as } s \to \infty.$$
This establishes the theorem.

4. FINAL VALUE THEOREM FOR HANKEL-CLIFFORD TRANSFORMATION

**Theorem 4.1** Let \( f(t) \) be a measurable function on \( 0 < t < \infty \) and if there exist a real \( \eta \) and a complex \( \alpha \) such that

\[
(i) \lim_{t \to \infty} \frac{f(t)}{t^\eta} = \alpha,
\]

and \( (ii) f(t)/t^\eta \) is bounded on \( 0 \leq t \leq y \) for all \( y > 0 \).

Then

\[
\lim_{s \to \infty} s^{\eta+1-\mu} C F(s) = \alpha,
\]

where \( F(s) \) and \( C \) are given as in **Theorem 3.1**; provided \( \eta + 1 > 0 \) and \( \mu > \eta - 1 \).

**Proof**: (4.1) indicates that \( f(t) \) is a function of slow growth and its Hankel-Clifford transform converges for \( s > 0 \). Proceeding as in the proof of **Theorem 3.1**, we have

\[
|s^{\eta+1-\mu} F(s) - \frac{\alpha}{C}| = |s^{\eta+1-\mu} \int_0^\infty s^{\mu} (st)^{-\mu/2} J_\mu (2\sqrt{s}t) f(t) \, dt - \alpha s^{\eta+1-\mu} \int_0^\infty s^{\mu} (st)^{-\mu/2} J_\mu (2\sqrt{s}t) \, dt| = |s^{\eta+1-\mu} \int_0^\infty s^{\mu} (st)^{-\mu/2} J_\mu (2\sqrt{s}t) [f(t) - \alpha t^\eta] \, dt| \leq \int_0^\infty |s^{\eta+1-\mu} s^{\mu} (st)^{-\mu/2} J_\mu (2st) [f(t) - \alpha t^\eta]| \, dt + \int_y^\infty |s^{\eta+1-\mu} s^{\mu} (st)^{-\mu/2} J_\mu (2st) [f(t) - \alpha t^\eta]| \, dt \]

\[ = J_1 + J_2.\]

From condition \( (i) \) of the theorem, we can find a small positive number \( \varepsilon \) such that for \( \eta \) large enough

\[
\sup_{y \leq t < \infty} |f(t)/t^\eta - \alpha| \leq \varepsilon C.
\]

Hence we have

\[
|J_2| \leq C \int_0^\infty s^{\eta+1-\mu} s^{\mu} (st)^{-\mu/2} J_\mu (2\sqrt{s}t) t^\eta \, dt = \varepsilon \quad \text{by \( (3.2) \)}
\]

hence

\[
|J_2| \leq \varepsilon.
\]
Therefore \( |J_2| \to 0 \) as \( \epsilon \to 0 \).

Now having fixed \( y \) as above, we have

\[
|J_1| \leq \sup_{0 \leq t \leq y} |f(t)/t^n| |\int_0^y |s^{\eta + 1 - \mu} s^\mu (st)^{-\mu/2} J_\mu (2\sqrt{st})| |dt.
\]

Due to conditions (ii) and (2.4), we have

\[
|J_1| \to 0 \text{ as } s \to \infty \text{ since } \eta + 1 > 0 \text{ and } \mu - \eta + 1 > 0.
\]

This completes the proof.

In extending this theorem to distributions we shall need the following result (Zemanian [Section 3.3]).

If \( f \in E'(I) \) there exists a constant \( C \) and a non-negative integer \( r \) such that for every \( \phi \in D(I) \),

\[
|<f, \phi>| \leq C \sup_{0 < t < \infty} |D_r^t \phi(t)|
\]

where \( D_r = \frac{d}{dt}^r \).

**Theorem 4.2 (distributional final value Abelian theorem).** If

(i) \( f \in H_{c,d} \) and \( f \) can be decomposed into \( f = f_1 + f_2 \) where \( f_1 \) is an ordinary function and \( f_2 \in E'(I) \), \( f_1 \) satisfies the hypothesis of Theorem 4.1.

(ii) \( f(t)/t^n \to \alpha \) as \( t \to \infty \) in the sense of Lojasiewicz.

(iii) \( 0 < \eta + 1 < \mu \);

(iv) \( F(s) \) is the distributional Hankel-Clifford transform of \( f \),

then

\[
\lim_{s \to \infty} s^{\eta + 1 - \mu} F(s) = \lim_{t \to \infty} \frac{f(t)}{t^n} C
\]

where \( C \) is defined in Theorem 3.1.

**Proof:** \( F(s) = F_1(s) + F_2(s) \) and

\( F_2(s) = <f_2(t), s^\mu (st)^{-\mu/2} J_\mu (2\sqrt{st})> \)

By theorem 2.2, is smooth function and is of slow growth as \( s \to \infty \). Let \( g(t) \in D(I) \) be identically equal to one on a neighbourhood of support of \( f_2 \).

From (4.2)

\[
|F_2(s)| = |<f_2(t), g(t)s^\mu (st)^{-\mu/2} J_\mu (2\sqrt{st})>|
\]

\[
\leq C_1 \sup_{0 < t < \infty} |D_r^t [g(t)s^\mu (st)^{-\mu/2} J_\mu (2\sqrt{st})]|
\]
\[= C_1 \sup_{0 < t < \infty} \sum_{v = 0}^{r} \left| D_r^{-v} g(t) \right| \left| D_l^v s^{\mu}(st)^{-\mu/2} J_{\mu} (2\sqrt{st}) \right|\]

\[= C_1 \sup_{0 < t < \infty} \sum_{v = 0}^{r} \left| D_r^{-v} g(t) \right| \left| (-1)^v s^{\mu + v} \right| \left| (st)^{-(\mu + v)/2} J_{\mu + v} (2\sqrt{st}) \right|\]

From (2.4) it is easy to see that

\[\left| (-1)^v s^{\mu + v} \right| \left| (st)^{-(\mu + v)/2} J_{\mu} (2\sqrt{st}) \right|\]

is bounded on \(0 < t < \infty\).

Hence for some constant \(C_2\),

\[\left| F_2(s) \right| \leq C_2\]

or

\[\left| s^{\eta + 1 - \mu} F_2(s) \right| \leq C_2 s^{\eta + 1 - \mu}\]

Under condition (iii),

\[\left| s^{\eta + 1 - \mu} F_2(s) \right| \to 0 \text{ as } s \to \infty.\]

Then

\[\lim_{s \to \infty} s^{\eta + 1 - \mu} F(s) = \lim_{s \to \infty} s^{\eta + 1 - \mu} F_1(s) + \lim_{s \to \infty} s^{\eta + 1 - \mu} F_2(s)\]

\[= \lim_{s \to \infty} s^{\eta + 1 - \mu} F_1(s).\]

Since \(f_1\) is an ordinary function which satisfies the hypothesis of Theorem 4.1 the required result follows.

REFERENCES


