Jñānābha, Vol.26,1996

# GENERALIZED LAPLACE TRANSFORM OF BANACH SPACE VALUED DISTRIBUTIONS

**Awadhesh Chandra Gupta** *and* **Anil Kumar Mahato**

*Department of Mathematics*
*K.C.B. College*
*Bero-835202, India*

*Department of Mathematics*
*Marwari College Ranchi*
*Ranchi-834001, India*

(*Received : April 20, 1993*)

## ABSTRACT

In this paper the generalized Laplace transform

$$F(s) = \frac{\Gamma(A)}{\Gamma(B)} \int_0^\infty (st)\ {}_1F_1\ (A, B;\ -st)\ f(t)dt$$

where $A = \beta + \eta + 1; B = \alpha + \beta + \eta + 1\ \beta \geq 0$ and $\eta > 0$ has been extended to certain Banach-space-valued distributions (generalized functions).

An inversion formula for the above transform is also proved in distributional (Banach space valued) sense.

## 1. INTRODUCTION

The theory of distributions has been worked out in great depth by L. Schwartz (4]. Sebastiaoe Sliva [5] also has a theory for such vector-valued distributions. Zemanian [7,8], has presented the theory of Banach space valued distributions. He has also discussed the Laplace transform of Banach-space-valued distributions. Further he has used these concept for applications in system theory and signals.

Various generalizations of the Laplace trnasform defined by the equation

$$F(s) = \int_0^\infty e^{-st} f(t)dt \qquad \dots (1.1)$$

have been studied by many mathematicians form time to time. One of such is given by

$$F(s) = \frac{\Gamma(A)}{\Gamma(B)} \int_0^\infty (st)\ {}_1F_1\ (A, B; -st)\ f(t)dt \qquad \dots (1.2)$$

where $A = \beta + \eta + 1; B = \alpha + \beta + n + 1; \beta \geq 0$ and $\eta = 0$. (1.2) reduces to (1.1) for $\alpha = \beta = 0$. In an earlier paper [3] a complex inversion formula for (1.2) has been extended to a class of generalized functions.

Motivated by the work of Zemanian [7, 8], we studied the transform defined by Eq. (1.2) for Banach-space-valued distributions. In this paper the complex inversion formula

$$\frac{f(t+)+f(t-)}{2} = \frac{1}{2\pi i}\int_{c-\infty}^{c+\infty} \frac{\Gamma(\alpha+n-s+1)t^{s-1}}{\Gamma(\beta+s)\,\Gamma(\eta-s+1)}\,\Phi(s)ds$$

where $\Phi(s) = \int_0^\infty x^{s-1} F(x)dx$, valid under certain conditions on $f(t)$ and parameters involved, has been extended to Banach-space-valued distributions for the trnasform (1.2).

The notation and terminology of this paper will follow that of Zemanian [7,8,9]. When $U$ and $V$ are two topological vector spaces, the symbol $[U; V]$ denotes the linear space of all continuous linear mappings of $U$ into $V$. Note that throught this work $D(A)$ will denote the space of smooth $A$-valued test functions with compact supports defined on $I = (0, \infty)$; $A$ and $B$ will denote Banach spaces, $R$ the set of all real numbers, and $C$ the set of all complex numbers.

## 2. The spaces $J_{c,d}(A)$ and $[J_{w,z}(A); B]$.

Given any two real numbers $c$ and $d$, $J_{c,d}(A)$ is defined as the linear space of all $A$-valued smooth functions $\phi(t)$ from $I$ into $A$ such that $i_k(\phi(t)) = i_{c,d,k}(\phi(t))$.

$$= \sup_{t\,\varepsilon\, I} \left| \lambda_{c,d}(t)\left(t\frac{d}{dt}\right)^{k} t\,\phi(t)\right|_A < \infty,\ k = 0, 1, 2, \ldots$$

where

$$\lambda_{c,d}(t) = \begin{cases} t^c, & 1 \le t < \infty \\ t^d, & 0 < t < 1. \end{cases}$$

The locally convex topology of $J_{c,d}(A)$ is defined by the family of seminorms $\{i_k\}_{k=0}^{\infty}$. For $A = C$ we write $J_{c,d}(A) = J_{c,d}$. It is easy to see that $D(A) \subset J_{c,d}(A)$ and consequently $D \subset J_{c,d}$.

Now let $\{c_j\}_{j=1}^{\infty}$ be a strictly decreasing sequence in $R$ tending to $w$ where either $w\ \varepsilon\ R$ or $w = \infty$. Similarly let $\{d_j\}_{j=1}^{\infty}$ be a strictly increasing sequence in $R$ tending to $z$, where either $z\ \varepsilon\ R$ or $z = \infty$.

The space defined by $J_{w,z}(A) = \bigcup_{j=1}^{\infty} J_{c_j,d_j}(A)$ is the inductive limit of the $J_{c_j,d_j}(A)$. It is a normal $\rho$-type testing function space.

Thus any continuous mapping $f$ of $J_{w,z}(A)$ into $B$ is called an $[A, B]$-valued generalized function on $I$.

For a Banach space $B$, any $f \varepsilon [D(A); B]$ will be called a Banach-space- valued distribution. Upon setting $A = C$, $f$ becomes a $B$-valued distribtuion and when $A = B = C$, $f$ becomes a scalar distribution. The simple of weaker topology for $[D(A);B]$ is generated by the collection of seminorms $\{\gamma_\phi\}_\phi$, where $\phi$ traverse $D(A)$ and

$$\gamma_\phi(f) = \|<f, \phi>\|_B.$$

We now come to the definition of $[J_{w,z}(A); B]$. If $\phi \varepsilon J_{w,z} : a \varepsilon A$, then $\phi a = a\phi$ is the function from $I$ into $A$ that assigns to each $t \varepsilon I$ the value $\phi(t)\, a$. Clearly $\phi a \varepsilon J_{w,z}(A)$. We denote by $J_{w,z}\, \Omega\, A$ the subspace of $J_{w,z}(A)$ consisting fo elements of the form $\phi a$. If $g \varepsilon [J_{w,z}; C]$ and $a \varepsilon A$, $ga = ag$ is defined by

$$< ga, \phi > = < g, \phi > a, \phi \varepsilon J_{w,z} \qquad \text{... (2.1)}$$

Clearly $ga \varepsilon [J_w, z\, ; A]$. $[J_{w,z}; C]\, \Omega\, \text{A}$ denotes the subspace of $[J_{w,z}; A]$ consisting of all linear combinations of elements of the form $ga$. Similarly if $g \varepsilon [J_{w,z}; [A; B]$ and $a \varepsilon A$, we define $ga$ by equ. (2.1) again. Now $ga \varepsilon [J_{w,z}\, ; B]$. $[J_{w,z}; [A; B]\, \Omega A$ denotes the subspace of $[J_{w,z}; B]$ consisting of all linear combinations of elements of the form $ga$. Thus every $f \varepsilon [J_{w,z}(A); B]$ uniquely defines a $g \varepsilon [J_{w,z}; [A; B]$ by means of the equation

$$< g, \phi > a = < f, \phi a > , \phi \varepsilon J_{w,z}, a \varepsilon A \qquad \text{... (2.2)}$$

## 3. GENERALIZED LAPLACE TRANSFORM

**Theorem 3.1.** For every fixed $s$, $\beta \geq 0$, $\eta > 0$,

$$k(st) = \frac{\Gamma(A)}{\Gamma(B)} (st)^\beta \; {}_1F_1(A, B; - st) \varepsilon J_{c,d}$$

when $c < \eta$ and $d > -(1 + \beta)$

**Proof :** From Erdelyi [1], p. 254, we have

$$\frac{d}{dx} [x^a \; {}_1F_1 (a, c; x)] = a\, x^{a-1} \; {}_1F_1 (a + 1, c; x) \qquad \text{... (3.1)}$$

$$\gamma_{c,d,k} [k(st)] = \sup_{t \varepsilon I} \| \lambda_{c,d}(t) \left(t \frac{d}{dt}\right)^k t\, k(st) \|_A$$

$$= \sup_{t \varepsilon I} \| \lambda_{c,d}(t) \left(t \frac{d}{dt}\right)^k t \frac{\Gamma(A)}{\Gamma(B)} (st)^\beta \; {}_1F_1(A, B; - st \|_A$$

$$= \sup_{t \varepsilon I} \|\lambda_{c,d}(t) \frac{\Gamma(A+k)}{\Gamma(B)} t(st)^{\beta} \; {}_1F_1(A+k, B; -st)\|_A$$

by (3.1)

From Slater [6], p.59, it follows that

$${}_1F_1(a, b-x) = o(1), x \to 0$$

and ${}_1F_1(a, b; -x) = \dfrac{\Gamma(b)}{\Gamma(b-a)} x^{-a} \{1 + o|x|^{-1}\}, x \to \infty$ Re $b >$ Re $a > 0$.

When $0 < t < 1$,

$$i_{c,d,k}[k(st)] = \sup_{t \varepsilon I} \| \frac{\Gamma(A+K)}{\Gamma(B)} t^{d+1} (st) \; {}_1F_1(A+k, B; -st)\|_A, \text{ w}$$

which is bounded as $x \to 0$ when $d > -(1+\beta)$.

Again, for $|t| \to \infty$,

$$i_{c,d,k}[k(st)] = \sup_{t \varepsilon I} = \| \frac{\Gamma(a+k)}{\Gamma(B)} t^{c+1} (st)^{\beta} \; {}_1F_1(A+k, B; -st)\|_A,$$

which is bounded when $c < \eta$. This completes our proof.

Now let $f \varepsilon [J_{w,z}; B]$, and we define the generalized Laplace transform $F(s)$ of $f$ by

$$F(s) = < f(t), k(st) >. \qquad \text{... (3.2)}$$

The left-hand side of $F(s)$ is a $B$-valued function. It can be shown as in [2] that $F(s)$ defined by (3.2) in an analytic function and

$$\frac{d}{ds} F(s) = <f(t), \frac{d}{ds} k(st)>.$$

Similarly if $y \varepsilon [J_{w,z}(A); B]$, the generalized Laplace transform $Y(s)$ of $y$ is defined as the generalized Laplace transform of $f_y$,

$$Y(s) = <f_y(t), k(st)> \qquad \text{... (3.3)}$$

where $f_y \varepsilon [J_{w,z}; [A; B]]$. The above definiton is meaningful because corresponding to $y$ we have a unique $f_y$ as discussed in the previous section. Note that $Y(s)$ is an $[A; B]$-valued function.

## 4. AN INVERSION THEOREM

We need the following three lemmas for the proof of inversion theorem.

**Lemma 4.1 :** Let $y \varepsilon [J_{w,z}(A); B], f_y$ be the corresponding member in $[J_{w,z}[A; B]]$ and

$$k(sx) = \frac{\Gamma(A)}{\Gamma(B)} (sx)^{\beta} \, {}_1F_1(A, B; -sx).$$

Then $\int_0^\infty x^{s-1} < f(u), k(xu)> dx = f(u), \int_0^\infty x^{s-1} h(xu)\, dx >$

**Lemma 4.2 :** Let $y \,\varepsilon\, [J_{w,z}(A); B]$, $\phi \,\varepsilon\, D(A)$ and

$$P(s) = \int_0^\infty t^{s-1} \phi(t)\, dt.$$

Then, for any two fixed real numbers $r$ and $c$ such that $0 < r < \infty$, $s = \sigma + iT$ with $c$ fixed and $c < \text{Re } s - 1 < d$, $-\infty < T < \infty$,

$$\frac{1}{2\pi} \int_{-r}^{r} < f_y(u), u^{-s} > \Gamma(s) dT$$

$$= < y(u), \frac{1}{2\pi} u^{-\varepsilon} P(s) dT >$$

**Lemma 4.3.** If $\phi \,\varepsilon\, D(A)$, then

$$\frac{1}{\pi} \int_0^\infty \phi(x) \left(\frac{x}{u}\right)^{\sigma - 1} [\sin r \log (x/u)][u \log (x/u)]^{-1} dx$$

tneds to $\phi(u)$ in $J_{w,z}(A)$ as $r \to \infty$

Except for some obvious changes, the proofs of these above lemmas are respectively the same as those of Lemmas 2.1, 2.2 and 2.3 in [3].

**Theorem 4.1.** If $y \in [J_{w,z}(A); (A); B]$, and let $Y(x)$ be the gneralized Laplace transform of Banach space-valued distribution y, then in the sense of weak convergence in $[D(A); B]$, for $c + 1 < \text{Re } s < d + 1$

$$\lim_{r \to \infty} < \frac{1}{2\pi i} \int_{\sigma - ir}^{\sigma + ir} \frac{\Gamma(\alpha + \eta - s + 1)}{\Gamma(\beta + s)\, \Gamma(\eta - s + 1)} M(s)\, t^{s-1} ds, \phi(t) >$$

$$= <y, \phi>$$

where $M(s) = \int_0^\infty x^{s-1} y(x)\, dx.$

**Proof.** On substituting

$$\frac{\Gamma(\alpha + \eta - s + 1)}{(\beta + S)\, \Gamma(\eta - s + 1)} = Q$$

$$= < \frac{1}{2\pi} \int_{-r}^{r} Q\, M(s) t^{s-1}\, ds, \phi(\tau) >. \qquad \ldots (4.1)$$

Since the integral on $s$ is an $A$-valued continuous function on $t$, it is a regular $A$-valued distribution and therefore the left-hand side can be written as

$$\frac{1}{2\pi} \int_{0}^{\infty} \phi(t) \int_{-r}^{r} QM(s) t^{s-1}\, dT\, dt,\ r > 0.$$

Since $y(x)\,\phi(t)$ is of bounded support and the integrand $Y(x)\phi(t)$is a continuous $B$-valued function of $(t, T)$, the order of integration may be changed to obtain

$$\frac{1}{2\pi} \int_{-r}^{r} Q\, M(s) \int_{0}^{\infty} \phi\,(t) t^{s-1}\, dt\, dT$$

That is,

$$\frac{1}{2\pi} \int_{-r}^{r} Q \,\{\int_{0}^{\infty} x^{s-1} < y(u), k(xu) > dx\} \int_{0}^{\infty} t^{s-1}\, \phi(t) dt\, dT$$

which by Lemma 4.1 is the same as

$$\frac{1}{2\pi} \int_{-r}^{r} Q\{< y(u), \int_{0}^{\infty} x^{s-1}\, k(xu)\, dx >\} \int_{0}^{\infty} t^{s-1}\, \phi(t)\, dt\, dT. \qquad \ldots (4.2)$$

Since from Erdelyi [1, p. 285], we have

$$\int_{0}^{\infty} \frac{\Gamma(\alpha + \eta - s + 1)}{\Gamma(\beta + s)\, \Gamma(\eta - s + 1)} x^{s-1}\, k(xu)\, dx = u^{-s},$$

(4.2) can be rewritten as

$$\frac{1}{2\pi} \int_{-r}^{r} < y(u), u^{-s} > \int_{0}^{\infty} t^{s-1}\, \phi(t)\, dt\, dT.$$

Again since $\phi$ is of bounded support and the integrand is a continuous $B$-valued function of $(t, T)$, the order of integration for the above repeated integral may be changed to obtain

$$< y(u), \frac{1}{2\pi} \int_{0}^{\infty} \phi\,(t) \int_{-r}^{r-s} u^{-s}\, t^{s-1}\, dT\, dt >. \qquad \ldots (4.3)$$

Since $\int_{-r}^{r-s} u\, t^{s-1}\, dT = 2(t/u)^{\sigma - 1}\, [u \log\,(t/u)]^{-1} \sin\,[r \log\,(\frac{t}{u})]$, (4.3) is the same as

$$< y(u), \frac{1}{\pi} \int_{0}^{\infty} \phi(t)\, (t/u)^{\sigma - 1}\, [u \log\,(t/u)]^{-1} \sin\,[r \log\,(t/u)] dx >. \qquad \ldots (4.4)$$

Now since $y \,\varepsilon\, [J_{w,z}(A); B]$ and by Lemma 4.3, the Banach-space-valued testing function in (4.4) converges to $\phi(U)$ in

$J_{w,z}(A)$, (4.4) tends to $<y, \phi>$ as $r \to \infty$, and that completes the proof.

## REFERENCES

[1] A. Erdelyi and Others - *Higher Transcendental Functions* Vol. I, Mc Graw Hill, 1953.

[2] A.C. Gupta and A.K. Mahato On a generalized Laplace transform of generalized functions. Communicated for publication in *Mathematika Balkanica.*

[3] A.C. Gupta and A.K. Mahato Complex inversion and uniqueness theorems for a generalized Laplace transform, *Indag. Math.*, N.S., **2(3)**, (1991), 301-310.

[4] L. Schwartz *Theorie des Distributions*, Hermann Paris, 1966.

[5] J.S. Silva, Sur la definition et la structure des distribution vectorielles, *Portugal. Math.*, **19**(1960), 1-80; Errata, 242-244.

[6] L.J. Slater, *Confluent Hypergeometric functions,* Cambridge Univ. Press, 1960.

[7] A.H. Zemanian The Hilbert Port, *SIAM J.Appl. Math.* **18** (1970), 98-138.

[8] A.H. Zemanian, *Realizability Theory for Continuous Linear Systems* Academic Press, New York, 1972.

[9] A.H. Zemanian *Generalized Integral Transformations,* Interscience, New York, 1968.