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SOME BILATERAL GENERATING RELATIONS INVOLVING HYPERGEOMETRIC FUNCTIONS OF TWO AND THREE VARIABLES

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ABSTRACT

In the present paper some bilateral generating relations on hypergeometric functions of two and three variables have been established. Some particular case, relevant to the prsent discussion are also discussed.

1. INTRODUCTION

If we use the notation

$$(a, n) = a(a+1)(a+2) \dots (a+n-1); \ (a, 0) = 1,$$

where a is arbitrary constant and n a positive integer, then the hypergeometric function of three variables $F_A, F_E, F_G, {}^3\phi_G^{(1)}$ have been defined by Lauricella [4], Saran [7], Sharma and Mittal [5], Jain [3]and Horn's functions of two variables $F_2, G_1, G_2, G_3, H_1, H_4, H_5, \phi_1, \psi_1$ as given in [1, p.224(7); p.224(10); p. 224(11); p.225(12); p.225(13); p.225(16); p.225(17); p.225(20); p.225(23)] are as follows :

$$F_A(\alpha, \beta_1; \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_3; x, y, z)$$

$$\sum_{m,n,p=0}^{\infty} \frac{(\alpha, m+n+p)(\beta_1, m)(\beta_2, n)(\beta_3, p)}{(1, m)(1, n)(1, p)(\gamma_1, m), (\gamma_2, n)(\gamma_3, p)} x^m y^n z^p, \ |x| + |y| + |z| < 1; \quad \dots (1.1)$$

$$F_E(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_3; z, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1, m+n+p)(\beta_1, m)(\beta_2, n+p)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n)(\gamma_3, p)} x^m y^n z^p, \ |x| < r, \ |y| < s, \ |z| < t, \quad \dots (1.2)$$

$r = (\sqrt{s} + \sqrt{t})^2 = 1;$

$F_G(\alpha_1, \alpha_1; \alpha_1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2; x, y, z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1, m+n+p)(\beta_1, m)(\beta_2, n)(\beta_3, p)}{(1, m)(1, n)(1, p)(\gamma_1, m)\,(\gamma_2, n+p)} x^m y^n z^p, \; |x| < 1, \; |y| < 1, \; |z| < 1; \quad \dots (1.3)$$

${}_3\phi_G^{(1)}\,(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_2; x, y, z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1, m+n+p)(\beta_1, m)}{(1, m)(1, n)(1, p)(\gamma_1, m)\,(\gamma_2, n+p)} x^m y^n z^p, \; |x| < 1, \; |y| < 1, \; |z| < 1; \quad \dots (1.4)$$

$$F_2(\alpha, \beta, \beta'; \gamma, \gamma'; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)(\beta, m)\,(\beta', n)}{(\gamma, m)(\gamma', n)\,(1, m)(1, n)} x^m y^n, \quad \dots (1.5)$$

$G_1(\alpha, \beta, \beta'; x, y)$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)(\beta, n-m)\,(\beta', m-n)}{(1, m)(1, n)} x^m y^n, \; |x| < r, \; |y| < r, \quad \dots (1.6)$$

$$r + s = 1;$$

$G_2(\alpha, \alpha', \beta, \beta'; x, y)$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha, m), (\alpha', n)(\beta, n-m)(\beta', m-n)}{(1, m)\,(1, n)} x^m y^n, \; |x| < 1, \; |y| < 1; \quad \dots (1.7)$$

$$G_3(\alpha, \alpha'; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha, 2n-m)(\alpha', 2m-n)}{(1, m)(1, n)} x^m y^n; \quad \dots (1.8)$$

$H_1(\alpha, \beta, \gamma, \delta; x, y)$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha, m-n)(\beta, m+n)(\gamma, n)}{(\delta, n)(1, m)(1, n)} x^m y^n, \; |x| < r, \; |y| < s, \quad \dots (1.9)$$

$$4rs = (s-1)^2;$$

$$H_4(\alpha, \beta; \gamma, \delta; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha, 2m+n)(\beta, n)}{(\gamma, m)(\delta, n)(1, m)(1, n)} x^m y^n, \; |x| < r, \; |y| < s \quad \dots (1.10)$$

$$4r = (s-1)^2;$$

$$H_5(\alpha, \beta; \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha, 2m+n)(\beta, m-n)}{(r, n)(1, m)(1, n)} x^m y^n,$$

$$|x| < r, \; |y| < s, \; 1 + 16r^2 - 36rs \pm (8r - s + 27rs^2) = 0; \quad \dots (1.11)$$

$$\phi_1(\alpha, \beta; \gamma; x, y) = \sum_{m, n=0}^{\infty} \frac{(\alpha, m+n)(\beta, n)}{(\gamma, m+n)(1, m)(1, n)} x^m y^n, \ |x| < 1; \quad .... (1.12)$$

$$\psi_1(\alpha,\beta;\gamma,\gamma'; x, y) = \sum_{m, n=0}^{\infty} \frac{(\alpha, m+n)(\beta, m)}{(\gamma, n)(\gamma', n)(1, m)(1, n)} x^m y^n, \ |x| < 1. \quad ... (1.13)$$

In the present investigations we require the following relations :

$$\sum_{n=0}^{\infty} \frac{(\lambda, n)}{(1, n)} F_1(-n, \mu\ \nu; \alpha; x, y)t^n = (1-t)^{-\lambda} F_1(\lambda, \mu, \nu; \alpha; \frac{xt}{t-1}, \frac{yt}{t-1}),$$

$$\max\{|xt/(t-1)|, |yt/(t-1)|, |t|\} < 1; \quad ... (1.14)$$

$$\sum_{n=0}^{\infty} \frac{(\lambda, n)}{(1, n)} F_2(-n, \nu, \gamma; \alpha, \beta; x, y)t^n = (1-t)^{-\lambda} F_2(\lambda, \mu, \gamma; \alpha, \beta, \frac{xt}{t-1}, \frac{yt}{t-1}),$$

$$\max\{|xt/(t-1)| + |yt/(t-1)|, |t|\} < 1; \quad ... (1.15)$$

$$\sum_{n=0}^{\infty} \frac{(\lambda, n)}{(1, n)} F_4(-n, \mu; \alpha, \beta; x, y)t^n = (1-t)^{-\lambda} F_4(\lambda, \mu; \alpha, \beta; \frac{xt}{t-1}, \frac{yt}{t-1}),$$

$$\max\{\sqrt{|xt/(t-1)|} + \sqrt{|yt/(t-1)|}, |t|\} < 1; \quad ... (1.16)$$

$$\sum_{n=0}^{\infty} \frac{(\lambda, n)}{(1, n)} t^n {}_{A+1}F_B\left[\begin{matrix} -n, (a); \\ & x \\ (b); \end{matrix}\right] = (1-t)^{-\lambda} {}_{A+1}F_B\left[\begin{matrix} \lambda, (a); \\ & \frac{xt}{t-1} \\ (b); \end{matrix}\right]; \quad ... (1.17)$$

where F_1, F_2, F_4 are Appels double hypergeometric function [1,p.224(6) (7), (9)]. Equations (1.14), (1.15), (1.16) are known results given by Srivastava [6, p. 86(4.1), (4.2) and (4.3)] and equation (1.17) is given by Chaundy [2, p.62(25)]..

(a) denotes the sequence of A parameters $a_1, a_2 \ldots a_A$ and $((a), m)$ is interpreted as $\prod_{j=1}^{A} (a_j, m)$.

2. BILATERAL GENERATING RELATIONS

We establish here the following bilateral generating relations:

$$\sum_{n=0}^{\infty} \frac{(\lambda, n)}{(1, n)} G_1(\lambda+n, \alpha, \alpha'; x, y) F_4(-n, \mu; \beta, \gamma; u, v)t^n$$

$$= (1-t)^{-\lambda} \sum_{q=0}^{\infty} \frac{(\lambda, q)(\alpha, q)}{(1-\alpha-p, q)(1, q)} (-\frac{y}{1-t})^q$$

$$F_E(\lambda+q, \lambda+q, \lambda+q, \alpha', \mu, \mu; 1-\beta-q, \beta, \gamma; \frac{-x}{1-t}, \frac{ut}{t-1}, \frac{vt}{t-1}); \quad \ldots (2.1)$$

$$\sum_{n=0}^{\infty} \frac{(\lambda, n)}{(1, n)} G_2(\lambda+n, \alpha', \beta, \beta'; x, y) F_1(-n, \mu, \gamma; \alpha; u, v)t^n$$

$$= (1-t)^{-\lambda} \sum_{q=0}^{\infty} \frac{(\alpha', q)(\beta, q)}{(1-\beta-p, q)(1, q)} (-y)^q$$

$$F_G(\lambda, \lambda, \lambda, \beta, \mu, \gamma; 1-\beta-q, \alpha, \alpha; \frac{-x}{1-t}, \frac{ut}{t-1}, \frac{vt}{t-1}); \quad \ldots (2.2)$$

$$\sum_{n=0}^{\infty} \frac{(\lambda, n)}{(1, n)} H_1(\lambda+n, \beta, \gamma, \delta; x, y) F_4(-n, \mu, \alpha, \beta; u, v)t^n$$

$$= (1-t)^{-\lambda} \sum_{q=0}^{\infty} \frac{(\beta, q)(\gamma, q)}{(1, q)(1-\lambda, q)} \{y(t-1)\}^q$$

$$F_E(\lambda-q, \lambda-q, \beta+q, \mu, \mu; \delta, \alpha, \beta; \frac{x}{1-t}, \frac{ut}{t-1}, \frac{vt}{t-1}); \quad \ldots (2.3)$$

$$\sum_{n=0}^{\infty} \frac{(\lambda, n)}{(1, n)} H_4(\lambda+n, \beta; \nu, \delta; x, y) F_2(-n, \mu, \nu; \alpha, \beta; u, v)t^n$$

$$= (1-t)^{-\lambda} \sum_{p=0}^{\infty} \frac{(\lambda, 2p)}{(1, p)(\gamma, p)} \left\{\frac{x}{(1-t)^2}\right\}^p$$

$$F_A(\lambda+2p, \beta, \mu, \nu; \delta, \alpha, \beta; \frac{y}{1-t}, \frac{ut}{t-1}, \frac{vt}{t-1}); \quad \ldots 2.4)$$

$$\sum_{n=0}^{\infty} \frac{(\lambda, n)}{(1, n)} H_5(\lambda+n, \alpha; \beta; x, y) F_1(-n, \mu, \gamma; \nu; u, v)t^n$$

$$= (1-t)^{-\lambda} \sum_{p=0}^{\infty} \frac{(\lambda, 2p)}{(1, p)(1-\alpha-q, p)} \left\{-\frac{x}{(1-t)^2}\right\}^p$$

$$F_G(\lambda+2p, \lambda+2p, \lambda+2p, \alpha, \mu, \nu; \beta, \gamma, \gamma; \frac{y}{1-t}, \frac{ut}{t-1}, \frac{vt}{t-1}); \quad \ldots (2.5)$$

$$\sum_{n=0}^{\infty} \frac{(\lambda, n)}{(1, n)} t^n G_1(\lambda+n, \beta, \beta'; x, y)\, {}_{A+1}F_B \begin{bmatrix} -n, & (a); \\ & & z \\ (b); & \end{bmatrix}$$

$$= (1-t)^{-\lambda} \sum_{r=0}^{\infty} \frac{(\lambda, r)((a), r)}{(1, r)((b), r)} [zt/(t-1)]^r$$

$$G_1(\lambda+r, \beta, \beta'; x(1-t), y(1-t)); \quad \ldots (2.6)$$

and

$$\sum_{n=0}^{\infty} \frac{(\lambda, n)}{(1, n)} t^n G_3(\alpha, \lambda+n; x, y)\, {}_{A+1}F_B\left[\begin{matrix} -n, (a); \\ \\ (b); \end{matrix} \; z\right]$$

$$= (1-t)^{-\lambda} \sum_{r=0}^{\infty} \frac{(\lambda, r)((a), r)}{(1, r)((b), r)} \left(\frac{zt}{t-1}\right)^r G_3(\alpha, \lambda+r, x(1-t)^2, \frac{y}{1-t}). \quad \text{... (2.7)}$$

To prove (2.1), consider

$$T = \sum_{n=0}^{\infty} \frac{(\lambda, n)}{(1, n)} G_1(\lambda+n, \alpha, \alpha'; x, y)\, F_4(-n, \mu; \beta, \gamma; u, v) t^n.$$

Expressing G_1 in series from as given in (1.6), employing elementary relation

$$(\lambda, n)\,(\lambda+n, p+q) = (\lambda, n+p+q) = (\lambda, p+q)\,(\lambda+p+q, n) \quad \text{... (2.8)}$$

and (1.16), we find

$$T' = \sum_{p,q=0}^{\infty} \frac{(\lambda, p+q)(\alpha, q-p)(\alpha, p-q)}{1, p)(1, q)}$$

$$x^p y^q (1-t)^{-\lambda-p-q} F_4(\lambda+p+q, \mu; \beta, \gamma; \frac{ut}{t-1}, \frac{vt}{t-1}). \quad \text{... (2.9)}$$

Further writing F_4 in series form and using a relation similar to (2.8) and

$$(\alpha, n-k) = \frac{(-1)^k (\alpha, n)}{(1-\alpha-n, k)},$$

$$T = (1-t)^{-\lambda} \sum_{q=0}^{\infty} \frac{(\lambda, q)\,(\alpha, q)}{(1, q)(1-\alpha-p, q)} \left(\frac{-y}{1-t}\right)^q$$

$$\sum_{p,r,s=0}^{\infty} \frac{(\lambda+q, p+r+s)(\alpha', p)(\mu, r+s)}{(1, p)(1, r)(1, s)(1-\alpha-q, p)(\beta, r)(r, s)} \left(\frac{-x}{1-t}\right)^p \left(\frac{ut}{t-1}\right)^r \left(\frac{vt}{t-1}\right)^s$$

which in the light of (1.2), provides (2.1).

The proof of the formulae (2.2), (2.3), (2.4) and (2.5) would run parallel to what we have obtained above.

To prove (2.6), consider

$$\Delta = \sum_{n=0}^{\infty} \frac{(\lambda, n)}{(1, n)} t^n G_1(\lambda+n, \beta, \beta'; x, y)\, {}_{A+1}F_B\left[\begin{matrix} -n \quad (a); \\ \\ (b); \end{matrix} \; z\right]$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda, n)}{(1, n)} t^n \sum_{p,q=0}^{\infty} \frac{(\lambda+n, p+q)(\beta, q-p)(\beta', p-q)}{(1, p)(1, q)} x^p y^q\, {}_{A+1}F_B\left[\begin{matrix} -n, (a); \\ \\ (b); \end{matrix} \; z\right].$$

If we employ (2.8) and use a relation similar to (1.17), we find

$$\Delta = (1-t)^{-\lambda} \sum_{p,q=0}^{\infty} \frac{(\lambda, p+q)(\beta, q-p)(\beta', p-q)}{(1,p)(1,q)} \left(\frac{x}{1-t}\right)^p \left(\frac{y}{1-t}\right)^q {}_{A+1}F_B \left[\begin{matrix} \lambda+p+q, (a); \\ \\ (b); \end{matrix} \; \frac{zt}{t-1}\right]$$

$$= (1-t)^{-\lambda} \sum_{p,q,r=0}^{\infty} \frac{(\lambda, p+q)(\beta, q-p)(\beta', p-q)((a), r)(\lambda+p+q, r)}{(1,p)(1,q)(1,r)((b),r)} \left(\frac{x}{1-t}\right)^p \left(\frac{y}{1-t}\right)^q \left(\frac{zt}{t-1}\right)^r$$

$$= (1-t)^{-\lambda} \sum_{r=0}^{\infty} \frac{(\lambda, r)((a),r)}{(1,r)((b),r)} \left(\frac{zt}{t-1}\right)^r \sum_{p,q=0}^{\infty} \frac{(\lambda+r, p+q)(\beta, q-p)(\beta', p-q)}{(1,p)(1,q)} \left(\frac{x}{1-t}\right)^p \left(\frac{y}{1-t}\right)^q$$

which in view of (1.6) provides (2.6).

The result (2.7) can be established exactly on the same lines as adopted above.

3. PARTICULAR CASES

(i) In (2.1), Putting $\beta = 0$ and simplifying, we have

$$\sum_{p,q=0}^{\infty} \frac{(\lambda, p+q)(\alpha, q-p)(\alpha', p-q)}{(1,p)(1,q)} \left(\frac{x}{1-t}\right)^p \left(\frac{y}{1-t}\right)^q {}_2F_1\left(\lambda+p+q, \mu; \beta; \frac{vt}{t-1}\right)$$

$$= \sum_{q=0}^{\infty} \frac{(\lambda, q)(\alpha', q)}{(1-\alpha, q)(1,q)} \left(\frac{-y}{1-t}\right)^q F_2\left(\lambda+q, \alpha'-q, \mu; 1-\alpha-q, \beta; \frac{-x}{1-t}, \frac{vt}{t-1}\right). \quad \text{... (3.1)}$$

(ii) Further changing $\frac{vt}{t-1}$ by $\frac{vt/(t-1)}{\mu}$ and taking limit as $\mu \to \infty$, (3.1) yields

$$G_1\left(\lambda, \alpha, \alpha', \frac{x}{1-t}, \frac{y}{1-t}\right) F\left(\lambda+p+q, \beta; \frac{vt}{t-1}\right)$$

$$= \sum_{q=0}^{\infty} \frac{(\lambda, q)(\alpha, q)}{(1-\alpha', q)(1,q)} \left(\frac{y}{1-t}\right)^q \psi_1\left(\lambda + q, \alpha-q, l-\alpha-q, \beta; \frac{x}{1-t}, \frac{vt}{t-1}\right). \quad \text{... (3.2)}$$

(iii) On taking $y = 0$ in (2.4) and solving, we obtain

$$\sum_{p,q=0}^{\infty} \frac{(\lambda, 2p+q)(p, q)}{(\gamma, p)(\delta, q)(1, p)(1, q)} \left[\frac{x}{(1-t)^2}\right]^p {}_2F_1\left(\lambda+2p+q, \mu; \beta; \frac{ut}{t-1}\right)$$

$$= \sum_{p=0}^{\infty} \frac{(\lambda, 2p)}{(\gamma, p)(1, p)} \left(\frac{x}{(1-t)^2}\right)^p F_2\left(\lambda+2p, \beta, \mu; \delta; \frac{y}{1-t}, \frac{ut}{t-1}\right). \qquad \ldots (3.3)$$

(*iv*) In (2.2) replacing v by $\frac{v}{\nu}$, taking the limit as $\nu \to \infty$ and letting $\alpha' = 0$, (2.2) provides

$$\sum_{p=0}^{\infty} \frac{(\lambda, p)(\beta', p)}{(1, p)(1-\beta-q, p)} \left(\frac{x}{1-t}\right)^p \phi_1\left(\lambda+p, \mu; \alpha; \frac{ut}{t-1}, \frac{vt}{t-1}\right)$$

$$= {}_3\phi_G^{(1)}\left(\lambda, \lambda, \lambda, \beta', \mu; 1-\beta-q, \alpha; \frac{x}{1-t}, \frac{ut}{t-1}, \frac{vt}{t-1}\right). \qquad \ldots (3.4)$$

REFERENCES

[1] Bateman Project, *Higher Transcendental Functions,* Vol. 1, McGraw-Hill, new York, (1953).

[2] T.W. Chaundy, An extension of hypergeometric functions, (I), *Quart. J. Math. (Oxford)* **14** (1943), 55-78.

[3] R.N. Jain, The confluent hypergeometric functions of three variables, *Proc. Nat. Acad. Sci. India.* **36** (A), II, (1966), 395- 408.

[4] G-Lauricella, Sulle funzini inpergeometric a piu variabili, *Rend. Circ. Mat. Palermo,* **7**, (1893), 111-158.

[5] B.L. Sharma and R.C. Mittal, Some new generating functions for Jacobi polynomials, *Camb. Phil. Soc.* **64,** (1968), 691-694.

[6] H.M. Srivastava, Certain formulas involving Appell functions, *Comment, Math. Univ. St. Pauli,* **22**(1), (1972), 73-79.

[7] S. Saran, Hypergeometric functions of three variables, *Ganita,* **5,** (1954), 77-91.