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SOLVABLE POTENTIALS FOR SCHLÄFLI'S CANONICAL FORM OF RICCATI'S AND MATHIEU EQUATIONS

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ABSTRACT

A few solvable potentials for the Schrödinger equation have been constructed by using Schläfli's canonical form of Riccati's and Mathieu differential equations following the method of Bhattacharjie and Sudarshan [1,p.864]. Similar potentials have been shown to be obtained by the method formulated by Bose [2, p.245]. Solution and S-Matrix of one of the potential derived from Schläfli's canonical form of Riccati's differential equation has also been discussed.

1. INTRODUCTION

Bhattacharjie and Sudarshan [1,p.864], using second order differential equation and transforming it to a form similar to Schrödinger equation, formulated a general method of constructing solvable potentials for the S-wave Schrödinger equation. For brevity, we shall write B.S. for the said authors. Bos [2, p.245], using the normal form of the second order differential equation and applying in the Schwarzian derivative, developed another method by which further solvable potentials were evaluated by him. In both these papers the potentials have been constructed by transforming the hypergeometric, confluent hypergeometric and Bessel differential equations.

The aim of present paper is to further evaluate a few solvable potentials by transforming Schläfli's canonical form of Riccati's and Mathieu differential equations by both B.S. method and Bose technique. Since the potentials obtained by both the methods are exactly similar, the fact that the techniques are in agreement is further established.

2. POTENTIALS FROM SCHLÄFLI'S CANONICAL FORM OF RICCATI'S EQUATION

B.S. Method

The Schlä.fli's canonical form of Ricati's equation [6,p. 90] is given by

$$\frac{du}{dt} = t^{\alpha} - t^{\alpha - 1}u^2. \qquad \text{... (2.1)}$$

To solve the equation, setting

$$u = t^{\alpha + 1}\,\frac{d(\log y)}{dt}. \qquad \text{... (2.2)}$$

Equation (2.1) reduces to

$$t\,\frac{d^2y}{dt^2} + (\alpha + 1)\,\frac{dy}{dt} - y = 0, \qquad \text{... (2.3)}$$

On making the substitutions

$$t = f(r),\ y(t) = g(r)\ \Phi\ (r),\ h(r) = \frac{d}{dr}\,[\log g(r)] \qquad \text{... (2.4)}$$

equation (2.3) can be transformed to the form

$$\Phi''(r) + A(r)\,\Phi'\,(r) + B(r)\,\Phi(r) = 0 \qquad \text{... (2.5)}$$

where

$$A(r) = 2\,\frac{g'(r)}{g(r)} - \frac{f''(r)}{f'(r)} + (1 + \alpha)\,\frac{f'(r)}{f(r)}, \qquad \text{... (2.6)}$$

$$B(r) = \frac{g''(r)}{g(r)} - \frac{\{f'(r)\}^2}{f(r)} - \frac{g'(r)}{g(r)}\left\{\frac{f''(r)}{f'(r)} - (1 + \alpha)\,\frac{f'(r)}{f(r)}\right\} \qquad \text{... (2.7)}$$

and primes denote differentiation.

Now for equation (2.5) to be of the form of s-wave radical Schrödinger equation (adopting units such that $E = k^2$)

$$\Phi''(r) + [k^2 - v(r)]\,\Phi(r) = 0 \qquad \text{... (2.8)}$$

and we should have

$$A(r) = 0,\ B(r) = k^2 - V(r)\ \text{ and }\ \frac{\partial}{\partial k}\,\{V(r)\} = 0. \qquad \text{... (2.9)}$$

Thus equations (2.6), (2.7) and (2.9) yield the following relations

$$f'(r) = Mg^2(r)\,\{f(r)\}^{1 + \alpha}, \qquad \text{... (2.10)}$$

where M is the integration constant, and

$$h'(r) - h^2(r) - \frac{\{f'(r)\}^2}{f(r)} \equiv k^2 - V(r). \qquad \text{... (2.11)}$$

Considering as particular choice

we obtain from (2.11)

$$\frac{1}{4}(1-\alpha^{2)}\,a^2e^{-2ar}\,(1-e^{-ar})^{-2}-a^2\,e^{-2ar}\,(1-e^{-ar})^{-1}-\frac{a^2}{4}\equiv k^2-v(r) \qquad \text{... (213)}$$

whcih suggest that

$$k^2=-\frac{a^2}{4} \qquad \text{... (2.14)}$$

$$V(r)=\frac{a^2(1-\alpha^2)}{4e^{2ar}\,(1-e^{-ar})^2}-\frac{a^2}{e^{2ar}\,(1-e^{-ar})} \qquad \text{... (2.15)}$$

Bose Method

The normal from of an ordinary differential equation in the notation of Bose [2,p.245] is written as

$$V''(t)+I(t)\,V(t)=0, \qquad \text{... (2.16)}$$

Putting $t=t(r)$ and $V(t)=\{t'(r)\}^{1/2}f(r)$, one gets

$$f''(r)+I_S(r)\,f(r)=0, \qquad \text{... (2.17)}$$

where

$$I_S=\{t'(r)\}^2\,I(t)+\frac{1}{2}\{t,r\}, \qquad \text{... (2.18)}$$

and $\{t, r\}$ is the Schwarzian derivative due to Erdélyi [3,p.96].

Now $I_S(r)$. of the S-wave radical Schrödinger equation is given by

$$I_S(r)=k^2-V(r), \qquad \text{... (2.19)}$$

where k^2 is purely parametric term.

For reducing the transformed Riccati's equation (2.3) to its normal form we apply the trnasformation

$$Y(t)=t^{-(\alpha+1)/2}\,W(t). \qquad \text{... (2.20)}$$

The transformed equation has the following form

$$W''(t)-\left|\frac{(\alpha^2+1)/4}{t^2}-\frac{1}{t}\right|W(t)=0. \qquad \text{... (2.21)}$$

Now $I(t)$ of the trnasformed Riccati's equation (2.3) is given from (2.21) as

$$I(t)=-\frac{(\alpha^2+1)}{4t^2}-\frac{1}{t}, \qquad \text{... (2.22)}$$

therefore, again for the choice $t=f(r)=\beta r$, using (2.18) and (2.22) we obtain

$$I_S(r) = -\frac{\alpha^2+1}{4r^2} - \frac{\beta}{r}, \qquad \text{... (2.23)}$$

and hence we finally arrive at the approximate potential given by Sharma and Singh [4,p.3 (*ii*)].

Similarly, for the choice $t = f(r) = 1 - e^{-\beta r}$, by using (2.18), (2.22) and (2.19) we are led to potential obtained earlier [4, p.3, eq. 14].

Solution and the *S*-matrix

We discuss here the solution and the *S*-matrix for the potential obtained in equation (2.15).

The general solution of Riccati's differential equation [6, p. 90] are as under

$$u = \frac{c_1 t^{\alpha+1} F(a+1, t) + c_2 F(-a-1, t)}{c_1 F(a, t) + c_2 t^{-\alpha} F(-a, t)}. \qquad \text{... (2.24)}$$

Thus the general solution $\Phi(r)$ of the Schrödinger equation (2.8) can be written as

$$\Phi(r) = \sqrt{\frac{M}{a}\frac{(1-e^{-ar})^{1+\alpha}}{e^{-ar}}}\ \frac{c_1(1-e^{-ar})^{\alpha+1} F(a+1, 1-e^{-ar})}{c_1 F(a, 1-e^{-ar}) + c_2(1-e^{-ar})^{-\alpha} F(-a, 1-e^{-ar})}$$

$$+ \sqrt{\frac{M}{a}\frac{(1-e^{ar})^{1+\alpha}}{e^{-ar}}}\ \frac{c_2 F(-a-1, 1-e^{-ar})}{c_1 F(a, l-e^{-ar}) + c_2(1-e^{-ar})^{-\alpha} F(-a, 1-e^{-ar})}$$

i.e.

$$\Phi(r) = \frac{P}{\sqrt{a}} e^{ikr}(1-e^{-2ikr})^{3(1+\alpha)/2} \frac{F(-a-1, 1-e^{-2ikr})}{c_1 F(a, 1-e^{-2ikr}) + c_2(1-e^{-2ikr})^{-\alpha}}$$

$$\frac{1}{F(-a, 1-e^{-2ikr})} + \frac{Q}{\sqrt{a}} e^{-ikr}(1-e^{2kir})^{3(1+\alpha)/2}$$

$$\times \frac{F(-a-1, 1-e^{2ikr})}{c_1 F(a, 1-e^{-2ikr}) + c_2(1-e^{-2ikr})^{-\alpha} F(-a, 1-e^{-2ikr})},$$

where the constant $\sqrt{M}$ being included in new constant P and Q. Constant P and Q to be determined so that $\Phi(0) = 0$ and $\Phi(r)$ is continum normalized.

Now

$$(1-e^{\pm 2ikr}) = (1-\cos 2kr) \pm i \sin 2kr.$$

Let

$$1 - \cos 2kr \sim \beta (r \to \infty)$$

and $\sin 2kr \sim \delta(r \to \infty)$,

then asymptotically,

$$\Phi(r) \sim N_1 e^{ikr} \frac{F(a+1, \beta + i\delta)}{c_1 F(a, \beta + i\delta) + c_2 (\beta + i\delta)^{-\alpha} F(-a, \beta + i\delta)}$$

$$+ N_2 e^{-ikr} \frac{F(-a-1, \beta - i\delta)}{c_1 F(a, \beta + i\delta) + c_2 (\beta + i\delta)^{-\alpha} F(-a, \beta + i\delta)}, \quad \text{... (2.26)}$$

where

$$N_1 = \frac{P}{\sqrt{a}} (\beta + i\delta)^{3/2\,(1+\alpha)}$$

and $$N_2 = \frac{Q}{\sqrt{a}} (\beta - i\delta)^{3/2(1+\alpha)},$$

It is to be noted that the value of β oscillates between 0 and 2 where $r \to \infty$ and it is likely to become 0 at some infinite points, only such points enable us to evaluate the S-matrix.

Thus, when β is zero, (2.26) takes the form

$$\Phi(r) \sim N_1 e^{ikr} \frac{F(a+1, i\delta)}{c_1 F(a, i\delta) + c_2 (i\delta)^{-\alpha} F(-a, i\delta)}$$

$$+ N_2 e^{-ikr} \frac{F(a-1, -i\delta)}{c_1 F(a, i\delta) + c_2 (i\delta)^{-\alpha} F(-a, i\delta)}, \quad \text{... (2.27)}$$

(2.27) together with the requirement $\Phi(r) = 0$ at $r = 0$, yields the S-matrix,

$$S(k) = -\frac{N_2}{N_1} = \frac{F(a+1, i\delta)}{F(-a-1, -i\,\delta)}. \quad \text{... (2.28)}$$

It should be noted, however that the above expression for the S-matrix is valid only for such values of β which tends to zero when $r \to \infty$.

3. POTENTIALS FROM MATHIEU EQUATION

B.S. Method

The Mathieu differential equation [5, p. 77] is written as

$$y''(z) + (n^2 - 2q \cos 2z)\, y(z) = 0. \quad \text{... (3.1)}$$

Now on making substitutions as given in (2.4), (3.1) assumes the form (2.5), where

$$A(r) = -\frac{f''(r)}{f'(r)} + \frac{2g'(r)}{g(r)}, \quad \text{... (3.2)}$$

$$B(r) = -\frac{g'(r)\, f''(r)}{g(r)\, f'(r)} + \frac{g''(r)}{g(r)} + (n^2 - 2q \cos 2f(r))\, \{f'(r)\}^2 \quad \text{... (3.3)}$$

and it will represent radial Schorödinger equation (2.8), if the conditions of (2.9) are satisfied.

Thus, we get

$$f'(r) = Mg^2(r)$$

where M is any arbitrary constant, and

$$h'(r) + h^2(r) + \{n^2 - 2q \cos 2f(r)\} \{f'(r)\}^2 \equiv k^2 - V(r). \quad \text{... (3.4)}$$

If we let $z = f(r) = 1 - e^{-ar}$, we obtain the potential

$$V(r) = -a^2 \{n^2 - 2q \cos 2(1 - e^{-ar})\} e^{-2ar}. \quad \text{... (3.5)}$$

On the other hand with the choice $z = f(r) = \beta r$

$$V(r) = 2q \cos 2\beta r, \quad \text{... (3.6)}$$

Bose Method.

Potential obtained in equation (2.29) can be rederived by this technique also.

$$I(z) = n^2 - 2q \cos 2z, \quad \text{... (3.7)}$$

thus,

$$I_s(r) = \beta^2 \{n^2 - 2q \cos 2\beta r\} \quad \text{... (3.8)}$$

and hence we finally arrive at the same result as obtained in (3.6).

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