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ON TACHIBANA CONCIRCULAR CURVATURE TENSOR

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ABSTRACT

The holomorphic projective curvature tensor and the Bochner curvature tensor of a Kaehler space correspond to the projective curvature tensor and the conformal curvature tensor respectively. There is a tensor in Kaehler space due to S. Tachibana, which corresponds to the concircular curvature tensor of the Riemannian space which we call the Tachibana concircular curvature tensor or T-concircular curvature tensor. The purpose of the present paper is to study the properties of this tensor.

1. INTRODUCTION

Let us consuder a $2n$-dimensional real manifold endowed with the almost complex structure F_i^h satisfying

$$F_i^h F_h^j = -\delta_i^j \qquad ...(1.1)$$

and a Hermitian metric

$$F_j^h F_i^l g_{hl} = g_{ij} \qquad ...(1.2)$$

such a space is called Hermit space.

Let us assume

$$F_{ij} = F_i^h g_{hj} \qquad ...(1.3)$$

then from (1.1), (1.2) and (1.3), we have

$$F_{ij} = -F_{ji} \qquad ...(1.4)$$

A Hermitian space in which

$$F_{i,j}^h = 0 \qquad ..(1.5)$$

and

$$F_{ij,k} = 0 \qquad ...(1.6)$$

where the *comma* (,) followed by an index denotes the operator of covariant differentiation with respect to the Riemannian metric g_{ij} is called a Kaehler space.

Let R^h_{ijk} be the curvature tensor of the kaehler space given by

$$R^h_{ijk} = \partial_i \left\{ \begin{matrix} h \\ jk \end{matrix} \right\} - \partial_j \left\{ \begin{matrix} h \\ ik \end{matrix} \right\} + \left\{ \begin{matrix} h \\ ia \end{matrix} \right\} \left\{ \begin{matrix} a \\ jk \end{matrix} \right\} - \left\{ \begin{matrix} h \\ ja \end{matrix} \right\} \left\{ \begin{matrix} a \\ ik \end{matrix} \right\} \quad ...(1.7)$$

where $\partial_i = \partial/\partial x^i$ and let us define

$$R_{ijkl} = R^h_{ijk}\, g_{hl} \quad ...(1.8)$$

where we have the following identities

$$R^h_{ijk} = -R^h_{jik} \quad ...(1.9)$$

and $$R_{ijkl} = -R_{jikl} = -R_{jkil} \quad ...(1.10)$$

The Ricci-tensor R_{ij} and the scalar curvature R are respectively given by

$$R_{ij} \overset{\text{def}}{=} R^h_{hij}$$

and $$R \overset{\text{def}}{=} R_{ij}\, g^{ij}.$$

The projective curvature tensor W^h_{ijk}, the conformal curvature tensor C^h_{ijk}, the concircular curvature tensor L^h_{ijk} and the conharmonic curvature tensor M^h_{ijk} are respectively given by

$$W^h_{ijk} = R^h_{ijk} + \frac{1}{2n-1}(R_{ik}\,\delta^h_j - R_{jk}\,\delta^h_i) \quad ...(1.11)$$

$$C^h_{ijk} = R^h_{ijk} + \frac{1}{2(n-1)}(R_{ik}\,\delta^h_j - R_{jk}\,\delta^h_i) + g_{ik}\,R^h_j$$

$$- g_{jk}\,R^h_i - \frac{R}{2(2n-1)(n-1)}(g_{ik}\,\delta^h_j - g_{jk}\,\delta^h_i) \quad ...(1.12)$$

$$L^h_{ijk} = R^h_{ijk} + \frac{R}{2n\,(2n-1)}(g_{ik}\,\delta^h_j - g_{jk}\,\delta^h_i) \quad ...(1.13)$$

and $$M^h_{ijk} = R^h_{ijk} + \frac{R}{2(n-1)}(g_{ik}\,R^h_j - g_{jk}\,R^h_i + R_{ik}\,\delta^h_j - R_{jk}\delta^h_i) \quad ...(1.14)$$

The tachibana concircular curvature tensor or the T-concircular curvature tensor T^h_{ijk} of the Kaehler space M_{2n} is given by (Tachibana, 1967)

$$T^h_{ijk} = R^h_{ijk} + \frac{R}{4n(n+1)} (g_{ik}\, \delta^h_j - g_{jk}\, \delta^h_i + F_{ik}\, F^h_j - F_{jk}\, F^h_i + 2F_{ij}\, F^h_k) \quad ...(1.15)$$

If we define

$$T^h_{ijkl} \overset{\text{def}}{=\!=} T^h_{ijk}\, g_{hl} \quad ...(1.16)$$

then (1.15) may be expressed as

$$T^h_{ijkl} = R_{ijkl} + \frac{R}{4n(n+1)} (g_{ik} g_{jl} - g_{jk} g_{il} + F_{ik} F_{jl} - F_{jk} F_{il} + 2F_{ij} F_{kl}) \quad ...(1.17)$$

Making use of (1.11), (1.12), (1.13), (1.14) and (1.15), the Tachibana concircular curvature tensor T^h_{ijk} may be expressed in terms of the projective curvature tensor, the conformal curvature tensor, the concircular curvature tensor and the conharmonic curvature tensor as follows

$$T^h_{ijk} = W^h_{ijk} + \frac{R}{4n(n+1)} (g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}\, F^h_j - F_{jk} F^h_i + 2F_{ij} F^h_k) - \frac{1}{2n-1} (R_{ik}\, \delta^h_j - R_{jk}\, \delta^h_i) \quad ...(1.18)$$

$$T^h_{ijk} = C^h_{ijk} + \frac{4n^2 - n + 1}{4n(2n-1)(n^2-1)} (g_{ik}\delta^h_j - g_{jk}\delta^h_i) + \frac{R}{4n(n+1)} (F_{ik} F^h_j - F_{jk} F^h_i + 2F_{ij} F^h_k) - \frac{1}{2(n-1)} (R_{ik}\delta^h_j - R_{jk}\delta^h_i + g_{ik} R^h_j - g_{jk} R^h_i) \quad ...(1.19)$$

$$T^h_{ijk} = L^h_{ijk} - \frac{3R}{4n(n+1)(2n-1)} (g_{ik}\delta^h_j - g_{jk}\delta^h_i) + \frac{R}{4n(n+1)} (F_{ik} F^h_j - F_{jk} F^h_i + 2F_{ij} F^h_k) \quad ...(1.20)$$

and

$$T^h_{ijk} = M^h_{ijk} + \frac{R}{4n(n+1)} (g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik} F^h_j - F_{jk} F^h_i + 2F_{ij} F^h_k) - \frac{R}{2(n-1)} (g_{ik} R^h_j - g_{jk} R^h_i + R_{ik}\delta^h_j - R_{jk}\delta^h_i) \quad ...(1.21)$$

We know that M_{2n} is a recurrent space if

$$R^h_{ijk,a} - \lambda a\, R^h_{ijk} = 0 \quad ...(1.22)$$

where λ_a is a non-zero recurrence vector. It is called Ricci-recurrent (or Semi-recurrent) space if

$$R_{ij,a} - \lambda_a R_{ij} = 0 \qquad ...(1.23)$$

where λ_a is a non-zero recurrence vector, which yields

$$R^h_{i,a} - \lambda_a R^h_i = 0 \qquad ...(1.24)$$

Multiplying (1.23) by g_{ij}, we obtain

$$R_{,a} - \lambda_a R = 0 \qquad ...(1.25)$$

Remark (1.1) *: Every recurrent space is Ricci-recurrent but the converse is not necesarily true.*

2. PROPERTIES OF TACHIBANA CONCIRCULAR CURVATURE TENSOR

Theorem (2.1) : *Tachibana concircular curvature tensor of a Kaehler space satisfies the Bianchi's first identity, that is*

$$T^h_{ijk} + T^h_{jki} + T^h_{kij} = 0 \qquad ...(2.1a)$$

and $$T_{ijkl} + T_{jkil} + T_{kijl} = 0 \qquad ...(2.1b)$$

Proof : Taking the cyclic permutation of the indices i, j and k in (1.15) and adding them, we have (2.1a). By virtue of (1.16), (2.1a) yield (2.1b).

Theorem (2.2) : *For a Kaehler space, the divergence of the curvature tensor, the Bochner curvature tensor and the Tachibana concicular curvature tensor are linearly dependent.*

Proof : Taking the divergence of (1.15), we have

$$\nabla . T^h_{ijk} = \nabla . R^h_{ijk} + \frac{\nabla.R}{4n(n+1)} (g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k) \qquad ...(2.2)$$

where $\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$.

It is given that (Tachibana, 1967)

$$\nabla . B^h_{ijk} = \frac{n}{n+2} W^h_{ijk} \qquad ...(2.3)$$

where B^h_{ijk} is the Bochner curvature tensor and

$$W^h_{ijk} = \nabla . R^h_{ijk} + \frac{\nabla.R}{4(n+1)} (g_{ik}\,\delta^h_j - g_{jk}\,\delta^h_i + F_{ik}\,F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k) \qquad ...(2.4)$$

Eliminating W^h_{ijk} between (2.3) and (2.4), we have

$$(n+2)(\nabla.B^h_{ijk}) - (n-1)(\nabla.R^h_{ijk}) - \nabla.T^h_{ijk} = 0 \qquad ...(2.5)$$

which proves the statement of the above theorem.

We, now, have the following :

Definition (2.1) : A space M_{2n} satisfying the condition

$$T^h_{ijk,a} - \lambda_a T^h_{ijk} = 0 \qquad ...(2.6)$$

for some non-zero recurrence vector λ_a, will be called a Tachibana concircular recurrent space or briefly a T-recurrent space.

Definition (2.2) : A space M_{2n} satisfying the condition

$$W^h_{ijk,a} - \lambda_a W^h_{ijk} = 0 \qquad ...(2.7)$$

for some non-zero recurrence vector λ_a, will be called a projective recurrent space or briefly a W-recurrent space.

Defintion (2.3) : A space M_{2n} satisfying the condition

$$C^h_{ijk,a} - \lambda_a C^h_{ijk} = 0 \qquad ...(2.8)$$

for some non-zero recurrence vector λ_a will be called a conformal recurrent or briefly a C-recurrent space.

Definition (2.4) : A space M_{2n} satisfying the condition

$$L^h_{ijk,a} - \lambda_a L^h_{ijk} = 0 \qquad ...(2.9)$$

for some non-zero recurrence vector λ_a, will be called a concircular recurrent space or briefly a L-recurrent spae.

Definition (2.5) : A space M_{2n} satisfying the condition

$$M^h_{ijk,a} - \lambda_a M^h_{ijk} = 0 \qquad ...(2.10)$$

for some non-zero recurrence vector λ_a, will be called a conharmonic recurrent space or briefly a M-recurrent space.

We have the following :

Theorem (2.3) : *For a Kaehler space, if any two of the following properties are satisfied, the third is also satisfied :*

(*i*) *it is a recurrent space,*

(*ii*) *it is a T-recurrent space,*

(*iii*) *it is a Ricci-recurrent space.*

Proof : Differentiation (1.15) covariantly with respect to x^a, we have

$$T^h_{ijk,a} = R^h_{ijk,a} + \frac{R_{,a}}{4n(n+1)}(g_{ik}\,\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}\,F^h_i + 2F_{ij}F^h_k) \quad ...(2.11)$$

Multiplying (1.15) by λ_a and subtracting the result thus obtained from (2.11), we get

$$T^h_{ijk,a} - \lambda_a\,T^h_{ijk} = R^h_{ijk,a} - \lambda_a\,R^h_{ijk} + \frac{(R_{,a} - \lambda_a R)}{4n(n+1)}(g_{ik}\delta^h_j - g_{jk}\,\delta^h_i + F_{ik}\,F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k) \quad ...(2.12)$$

Suppose that the Kaehler space is recurrent and T-recurrent, so that the equations (1.22) and (2.6) are satisfied. Now (2.12), in view of (1.22) and (2.6) reduced to

$$\frac{(R_{,a} - \lambda_a R)}{4n(n+1)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}\,F^h_j - F_{jk}\,F^h_i + 2F_{ij}F^h_k) = 0 \quad ...(2.13)$$

which after some simplification gives (1.25) and this implies that the space is Ricci-recurrent space. The proofs of the other parts of the theorem follow the same pattern.

Similarly, we can prove the following theorems by taking into account the equations (1.22), (1.23), (1.24), (1.25) and definitions (2.1), (2.2), (2.3), (2.4) and (2.5).

Theorem (2.4) : *If a Kaehler space satisfies any two of the following properties :*

(*i*) *it is a Ricci-recurrent space*

(*ii*) *it is a T-recurrent space*

(*iii*) *it is a W-recurrent space*

then it must also satisfy the third.

Theorem (2.5) : *If a Kaehler space satisfies any two of the following properties :*

(*i*) *it is a Ricci-recurrent space*

(*ii*) *it is a T-recurrent space,*

(*iii*) *it is a C-recurrent space*

then it must also satisfy the third.

Theorem (2.7) : *If a Kaehler space satisfies any two of the following properties :*

(i) it is a Ricci-recurrent space

(ii) it is a T-recurrent space

(iii) it is a M-recurrent space

then it must also satisfy the third.

Theorem (2.8) : *A necessary and sufficient condition for a T-recurrent spce to be recurrent is that the spae be Ricci- recurrent.*

Proof : Suppose that the T-recurrent space is a Ricci-recurrent space, so that (1.23) and (2.6) are satisfied. Now, (2.12) in view of (1.25) and (2.6) reduces to

$$R^h_{ijk,a} - \lambda_a R^h_{ijk} = 0$$

which shows that the space is recurrent space this completes the proofs.

REFERENCES

[1] S. Tachibana : On the Bochner curvature tensor, *Natural Sci. Rep. Ochanomizu Univ*. Vol. **18**(1) (1967), 15-19.

[2] K. Yano : *Differential Geometry on Complex and Almost Complex Spaces.* Pergamon Press, New York, (1965).