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THERMAL STABILITY OF NON-HOMOGENEOUS VISCOUS COUETTE FLOW

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ABSTRACT

Here the stability of an incompressible, non-homogeneous, viscous and heat conducting fluid flow between two rotating coaxial circular cylinders in presence of axial temperature gradient has been discussed. Sufficient condition for stability has been obtained in different cases. A condition is also obtained under which oscillatory modes are stable.

1. INTRODUCTION

The stability of couette flow between rotating coaxial circular cylinders has been studied extensively by many authors. Taylor [5] observed both theoretically and experimentally that at a sufficient fast rotation of the inner cylinder the flow of an incompressible, viscous, homogeneous non-conducting fluid between two coaxial rotating circular cylinders become unstbale. Banerjee and Pradhan [1] considered the effect of axial, gravitational force on the stability of the system in presence of an axial non- homogeneity and showed that if the density variation decreases in the axial direction then the effect of this decrease in density is stabilising. Saini and Agrawal [3] discussed the effect of an axial temperature gradient in the presence of an axial gravitational force on the stability of a viscous, incompressible and heat conducting fluid between two coaxially rotating circular cylinders of infinite lengths. They showed that the negative temperature gradient has a destabilising character while the positive temperature gradient has stabilising effect.

In the present paper, we have discussed the stability of an incompressible, non-homogeneous, viscous and heat conducting fluid flows between two rotating coaxial circular cylinders in presence of axial temperature gradient. We have obtained sufficient conditions for stability in different cases. We have also obtained condition under which only non-oscillatory modes may exist.

2. FORMULATION

Let an incompressible, heat conducting and viscous fluid be flowing between two infinitely long coaxial cylinders of radii R_1 and R_2 ($>R_1$). Let the inner and outer cylinders be rotating with angular speeds Ω_1 and Ω_2 respectively. Let the basic state be given by

$$u_r = U_z = 0 \text{ and } U_\theta = V(r) = r\Omega(r) = Ar + B/r^2 \qquad ...(1)$$

together with

$$\rho = \rho(z)[1 + \alpha(T - T_0)], \frac{\partial p}{\partial r} = \frac{V^2}{r} \frac{\partial p}{\partial z} = g\rho \text{ and } \frac{\partial T}{\partial z} = \beta \qquad ...(2)$$

where

$$A = \Omega_1 \frac{(\mu - \eta^2)}{(1 - \eta^2)}, B = \Omega_1 R_1^2 \frac{(1 - \mu)}{(1 - \eta)^2}, \mu = \frac{\Omega_2}{\Omega_1}, \eta = \frac{R_1}{R_2},$$

λ is a real parameter, ρ_0 is a positive constant, ρ and p are density and pressure at any point in the flow domain respectively and g is the acceleration due to gravity.

Let us now consider that the basic state is slightly perturbed. Writing down the equation of continuity, equation of motion, incompressibility and energy equation in cylindrical coordinates, linearising them for small perturbations expressed into normal modes as

$$f'(r, z, t) = f(r) \exp(ikz + nt),$$

where n is a complex constant and k is real. Eliminating various physical quantities from the linearised perturbation equations using narrow gap approximation, the final stability governing equations are

$$(D^2 - a^2 - \sigma)(D^2 - a^2)U_r + \frac{R_a}{\sigma} D^2 U_r = -a^2 \tau(1 + \alpha x)U_\theta + R_b a)\phi \qquad ..(3)$$

$$(D^2 - a^2 - \sigma p)\phi = \frac{DU_r}{a}, \qquad ...(4)$$

$$(D^2 - a^2 - \sigma)U_\theta = U_r, \qquad ...(5)$$

where $R_a = \dfrac{\lambda g d^4}{v^2}$, $R_b = \dfrac{\alpha\beta g d^4}{K}$, $x = \dfrac{r - R_1}{d}$, $a = (K - d)$, $\sigma = \dfrac{nd^4}{\eta}$

and $\alpha = (\mu - 1)$.

Here τ is the Taylor number, R_a is the static stratification number and R_b is the themral stratification number.

The boundary conditions are

$$U_r = U_\theta = dU_r = \phi = 0 \text{ at } x = 0 \text{ and } 1. \qquad \text{... (6)}$$

3. CASE I : AVERAGE VALUE OF $(1 + \alpha x)$:

(a) **Sufficient Condition of Stability :** The average value of $(1 + \alpha x)$ will be $\left(\frac{1+\mu}{2}\right)$. Now putting this value of $(1 + \alpha x)$ in the equation (3), we have

$$(D^2 - a^2 - \sigma)(D^2 - a^2)U_r + \frac{R_a}{\sigma} D^2 U_r = -a^2\tau\left(\frac{1+\mu}{2}\right)U_\theta + R_b a D\phi \qquad \text{...(7)}$$

Multiplying equation (7) by $\overline{U}_r$, integrating over the range of x and using equation (4), (5) and boundary conditions (6), we have

$$\int [|D^2U_r|^2 + 2a^2|DU_r|^2 + a^4|U_r|^2 + \sigma\int[|dU_r|^2 + a^2|U_r|^2] - \frac{R_a}{\sigma}\int |dU_r|^2$$

$$= a^2\tau\,(\frac{1+\mu}{2})\,[\int \{|DU_\theta|^2 + a^2|U_\theta|^2 + \overline{\sigma}\int |U_\theta|^2]$$

$$+ a^2R_b[\int \{|D\phi|^2 + a^2|\phi|^2\} + \overline{\sigma}p\int |\phi|^2]. \qquad \text{...(8)}$$

The real and imaginary parts of equation (8) are

$$I_1 - a^2\tau\,(\frac{1+\mu}{2})\,I_4 - a^2R_bI_6$$

$$+ \sigma_r[I_2 - \frac{R_a}{|\sigma|^2}I_3 - a^2\tau\,(\frac{1+\mu}{2})\,I_5 - a^2\,pR_bI_7] = 0 \quad \text{...(9)}$$

and

$$\sigma_i\,[I_2 + \frac{R_a}{|\sigma|^2}I_3 + a^3\tau\,(\frac{1+\mu}{2})\,I_5 + a^2pR_bI_7] = 0 \qquad \text{...(10)}$$

where $I_1 = \int [|D^2U_r|^2 + 2a^2|DU_r|^2 + a^4|U_r|^2]$,

$I_2 = \int [|dU_r|^2 + a^2\;|U_r|^2]$,

$I_3 = \int |DU_r|^2$,

$I_4 = \int [DU_\theta|^2 + a^2\;|U_\theta|^2]$,

$I_5 = \infty\;|U_\theta|^2$,

$I_6 = \int [|D\phi|^2 + a^2 |\phi|^2]$,

$I_7 = \int |\phi|^2$.

Theorem 1. If $\tau < 0$ (Co-rotating cylinders), $R_a < 0$, $R_b < 0$ and $\mu > -1$ then system is stable.

Proof. If $\tau < 0$, $R_a < 0$, $R_b < 0$ and $\mu > -1$ then for the consistency of equation (9) it is necessary that $\sigma_r < 0$ implying the stability of the system.

Theorem 2. If $\tau > 0$ (cylinders moving in opposite directions), $R_a < 0$, $R_b < 0$ and $\mu < -1$ then the system is stable.

Proof. If $\tau > 0$, $R_a < 0$, $R_b < 0$ and $\mu < -1$ then for the consistency of equation (9) it is necessary that $\sigma_r < 0$ implying the stability of the system.

Theorem 3. If $\tau > 0$, $R_a > 0$, $R_b > 0$ and $\mu > -1$, then only non-oscillatory modes may exist.

Proof. If $\tau > 0$, $R_a > 0$, $R_b > 0$ and $\mu > -1$ then expression (10) implies that $\sigma_i = 0$ i.e. under these conditions only non-oscillatory modes may exist. In other words we may say that under the above conditions oscillatory modes will be absent.

(*b*) On the existence of oscillatory perturbations, we derive

Theorem 4. If $\tau < 0$ and $R_b > 0$, then oscillatory modes are stable.

Proof. For $\sigma_i \neq 0$ expression (10) reduces to

$$I_2 + \frac{R_a}{|\sigma|^2} I_3 + a^2 \tau \left(\frac{1+\mu}{2}\right) I_5 + a^2 p R_b I_7 = 0. \quad ...(11)$$

Now from equations (9) and (11), we may have

$$I_1 - a^2 \tau \left(\frac{1+\mu}{2}\right) I_4 + a^2 R_b I_6 + 2\sigma_r [I_2 + a^2 p R_b I_7] = 0. \quad ...(12)$$

If $\tau < 0$ and $R_b > 0$, then for the consistency of equation (12) it is necessary that $\sigma_r < 0$ implying the stability of the system. This shows that under the above conditions oscillatory modes will be stable.

4. CASE II : GENERAL CASE

Theorem 5. If $\tau < 0$, $R_a < 0$, $R_b < 0$ and $\alpha > 0$ (outer cylinder moving faster than the inner cylinder), then the system is stable.

Proof. Multiplying equation (3) by $\overline{U}_r$ integrating over the range of x and using equations (4), (5) and boundary conditions (6), we have

$$\int[|D^2U_r|^2 + 2a^2|DU_r|^2 + a^4|U_r|^2] + \sigma\int[|DU_r|^2 + a^2|U_r|^2] - \frac{R_a}{\sigma}\int|DU_r|^2$$
$$= a^2\tau\int(1+\alpha x)[|DU_\theta|^2 + a^2|U_\theta|^2 + \overline{\sigma}\,|U_\theta|^2] + a^2\tau\alpha\int U_\theta D\overline{U}_\theta$$
$$+ a^2R_b\int[|D\phi|^2 + a^2\,|\phi|^2 + \overline{\sigma}p\,|\varphi|^2]. \qquad ...(13)$$

The real part of (13) gives

$$I_1 - a^2\tau\int(\frac{\alpha}{2a} + 1 + \alpha x)[|DU_\theta|^2 + a^2|U_\theta|^2 + \sigma_r[I_2 - \frac{R_a}{|\sigma|^2}I_3$$
$$- a^2\tau\int(1+\alpha x)\,|U_\theta|^2 - a^2pR_b\int|\phi|^2 - a^2R_bI_6 \le 0 \qquad ...(14)$$

Now if $\tau < 0$, $R_a < 0$, $R_b < 0$ and $\alpha > 0$ then for the consistency of equation (14) it is necessary that $\sigma_r < 0$ implying the stability of the system.

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